UKMT British Mathematical Olympiad Round 2 Unofficial Solutions

Krish Nigam

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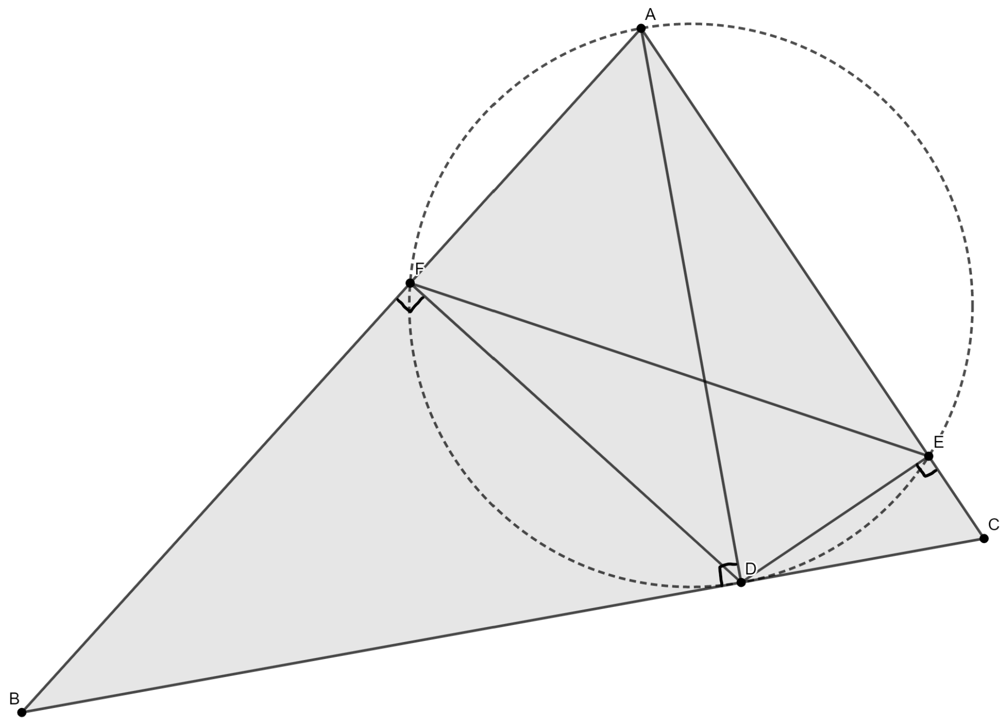
1 BMO2 2002

1.1 Problem 1

The altitude from one of the vertices of an acute-angled triangle ABC meets the opposite side at D. From D perpendiculars DE and DF are drawn to the other two sides. Prove that the length of EF is the same whichever vertex is chosen.

Solution. We want to show that the length EF is independent of which vertex created it. I have arbitrarily chosen vertex A.



Let the angles at vertex A, B, C be equal to $\angle A$, $\angle B$, $\angle C$ respectively.

We know $\angle EDC = 90° - \angle C \implies \angle ADE = \angle C$.

Since $\angle DEA = 90°$ and $\angle AFD = 90°$, $\angle DEA + \angle AFD = 180°$. Therefore, quadrilateral AFDE is cyclic and we can circumscribe.

By angles in the same segment, $\angle ADE = \angle AFE = \angle C$. Now, we can use the sine rule in $\triangle AFE$, in order to find an expression for EF.

$$\frac{EF}{\sin \angle A} = \frac{AE}{\sin \angle C} \implies EF = AE \cdot \frac{\sin \angle A}{\sin \angle C} \tag{1}$$

However, we want to show that this expression does not change due to the vertex picked. Let's consider the area of the total triangle $[ABC]$ and its circumradius R. By the full sine rule:

$$\frac{AB}{\sin \angle C} = \frac{BC}{\sin \angle A} = \frac{CA}{\sin \angle B} = 2R$$

By the area of a triangle formula '$\frac{1}{2}ab\sin\theta$', we see:

$$\begin{aligned}[ABC] &= \frac{1}{2}AB \cdot BC \cdot \sin\angle B \\ &= \frac{1}{2}(2R\sin\angle C)\cdot(2R\sin\angle A)\cdot\sin\angle B \\ &= 2R^2\sin\angle A\cdot\sin\angle B\cdot\sin\angle C\end{aligned}$$

In (1), we established that $EF = AE \cdot \frac{\sin\angle A}{\sin\angle C}$. So, I want to find an expression for AE.

$$\begin{aligned}AE = AC - EC\,,\ \ EC &= CD\cdot\cos\angle C \\ &= (AC\cos\angle C)\cdot\cos\angle C \\ &= AC\cos^2\angle C\end{aligned}$$

$$AE = AC - AC\cos^2\angle C = AC\left(1-\cos^2\angle C\right) = AC\sin^2\angle C$$
$$\implies EF = AC\sin\angle A\cdot\sin\angle C$$

By the full sine rule though, $AC = 2R\sin\angle B$.

$$\therefore\ EF = 2R\sin\angle A\cdot\sin\angle B\cdot\sin\angle C = \boxed{\frac{[ABC]}{R}}$$

Hence, the length EF is independent of the vertex chosen since it only depends on the area of the triangle and its circumradius which are constants pre-defined to the triangle itself. □

1.2 Problem 2

A conference hall has a round table with n chairs. There are n delegates to the conference. The first delegate chooses his or her seat arbitrarily. Thereafter the $(k+1)$th delegate sits k places to the right of the kth delegate, for $1 \leq k \leq n-1$. No chair can be occupied by more than one delegate.

Find the set of values n for which this is possible.

Solution. In this problem, we have n chairs so it will make sense to deal with things modulo n. This is because after going round all n seats you will end up back at the original seat and the counting resets.

If we label our seats $0, 1, \ldots, n-1$ going counterclockwise (*i.e.* in the right direction, as per the question), then we have:

- WLOG, the 1st person sits in seat 0
- the 2nd person sits in seat 1
- the 3rd person sits in seat 3
- and the kth person sits in seat $\equiv \frac{k(k+1)}{2} \pmod{n}$

This is because the kth person sits $0 + 1 + 2 + \cdots + (k-1)$ seats to the right of the first person.

We should begin by considering what happens for small values of n. If $n = 1$ and $n = 2$ then clearly this works. When $n = 3$, both the first and the third person will end up in the same seat which is not good. Our seating however works when $n = 4$. Exploring further, we should find that the next time this works is for $n = 8$. From this, I see a pattern.

I CONJECTURE that the set of values n for which the seating is possible is when $n = 2^k$ for all non-negative integers k.

It is often a good idea to see where our system goes wrong and how this can narrow down our possibilities for the values of n. For some n, we will have a problem if there are two people who have the same seat. Going back to the numbering of the seats above, there cannot exist two different numbers p, q with $1 \leq p < q \leq n-1$ such that:

$$\frac{p(p-1)}{2} \equiv \frac{q(q-1)}{2} \pmod{n} \tag{2}$$

While it is tempting to divide through by $\frac{1}{2}$ in our above expression, it is better to pause and make use of an important idea. If $a \equiv b \pmod{n} \iff ka \equiv kb \pmod{kn}$ for all real k.[1] This idea instead enables us to make much greater progress since we are then dealing in modulo $2n$ which gives us more tools than merely dealing in modulo n.

Multiplying (2) by 2 gives

$$p(p-1) \equiv q(q-1) \pmod{2n}$$

[1] There is also another rule when k is an integer that $a \equiv b \pmod{n} \iff ka \equiv kb \pmod{n}$, but this isn't so helpful here.

Now, $q(q-1) - p(p-1) = (q-p)(p+q-1)$ so we have a clash of seating if and only if

$$(q-p)(p+q-1) \equiv 0 \pmod{2n} \tag{3}$$

Let's suppose that $2n$ is in fact a power of 2 (based on our conjecture). Let's also suppose, for contradiction, that there are two different seats p, q with $p < q$ which end up being occupied. So, from (3), that means $2n$ is a factor of $(q-p)(p+q-1)$.

But, observe that *exactly one* of $p-q$ or $p+q-1$ can be even. Therefore, $2n$ will divide either of $p-q$ or $p+q-1$, but since $p-q < n$ it must be that $2n$ divides $p+q-1$. However, $p+q-1 < 2q-1 < 2n$ as $q \leq n$. So, $p+q-1 < 2n$ and so $2n$ cannot divide $p+q-1$. Hence, there exists no such different values of p, q.

But that only shows $n = 2^k$ is a possible candidate for values of n. We also want to prove that powers of 2 are the *only* solutions for n.

So, suppose for contradiction that $2n = xy$ is not a power of 2. This means that one of x or y is odd and the other is a power of 2. Without loss of generality, suppose that $x < y$. Now, we can manufacture the incident of our seating going wrong as per (3). We can pick appropriate values for p and q by considering $(q-p)(p+q+1) = 2n$. We can take $p = \frac{y-x+1}{2}$ and $q = \frac{x+y+1}{2}$; we can do this because it satisfies $p < q$ and both $\frac{y-x+1}{2}$ and $\frac{x+y+1}{2}$ are integers due to only one of x, y being odd as established.

Therefore, there is nothing stopping us having someone in seat $p = \frac{y-x+1}{2}$ and someone else in seat $q = \frac{x+y+1}{2}$. Consider $(q-p)(p+q-1)$:

$$\begin{aligned}(q-p)(p+q-1) &= \left(\frac{x+y+1}{2} - \frac{y-x+1}{2}\right)\left(\frac{y-x+1}{2} + \frac{x+y+1}{2} - 1\right) \\ &= (x)(y) \\ &= 2n\end{aligned}$$

But, this leads to (3) as $(q-p)(p+q-1) \equiv 0 \pmod{2n}$. We can't have this occur!

Hence, if $2n$ is not a power of 2, then we break down. This means that n must be a power of 2. □

1.3 Problem 3

Prove that the sequence defined by

$$y_0 = 1, \quad y_{n+1} = \frac{1}{2}\left(3y_n + \sqrt{5y_n^2 - 4}\right), \quad (n \geq 0)$$

consists only of integers.

Solution. First let's find an explicit formula for y_{n-1} in terms of y_n. Our sequence can also be written as:

$$y_n = \frac{1}{2}\left(3y_{n-1} + \sqrt{5y_{n-1}^2 - 4}\right) \quad \text{for } n \geq 1$$

$$\therefore\ 2y_n - 3y_{n-1} = \sqrt{5y_{n-1}^2 - 4}$$

$$\implies 4y_n^2 + 9y_{n-1}^2 - 12y_{n-1}y_n = 5y_{n-1}^2 - 4$$

$$4y_n^2 + 4y_{n-1}^2 - 12y_{n-1}y_n + 4 = 0$$

$$\therefore\ \underline{y_{n-1}^2 - 3y_n \cdot y_{n-1} + \left(y_n^2 + 1\right) = 0}$$

So, what we have above now is a quadratic in y_{n-1}. So, solving this gives

$$y_{n-1} = \frac{3y_n \pm \sqrt{9y_n^2 - 4\left(y_n^2 + 1\right)}}{2} = \frac{1}{2}\left(3y_n \pm \sqrt{5y_n^2 - 4}\right)$$

But, we must REJECT $y_{n-1} = \frac{1}{2}\left(3y_n + \sqrt{5y_n^2 - 4}\right)$, since this would imply that $y_{n-1} = y_{n+1}$ for all n, but that's clearly false by counterexample:

$$(n = 1) \quad y_0 = 1,\ y_2 = 5 \implies y_0 \neq y_2$$

Hence, it must be that $y_{n-1} = \frac{1}{2}\left(3y_n - \sqrt{5y_n^2 - 4}\right)$. Now, consider y_{n+1} from the question.

$$y_{n+1} = \frac{1}{2}\left(3y_n + \sqrt{5y_n^2 - 4}\right) = 3y_n - \frac{1}{2}\left(3y_n - \sqrt{5y_n^2 - 4}\right) = 3y_n - y_{n-1}$$

$$\therefore\ \underline{y_{n+1} = 3y_n - y_{n-1}} \quad \text{for all } n \geq 1$$

By this, it is clear that all members of the sequence are integers, provided that the first two are integers.

Checking the first two terms: $y_0 = 1, y_1 = 1, y_2 = 3(1) - 1 = 2 \in \mathbb{Z}$. □

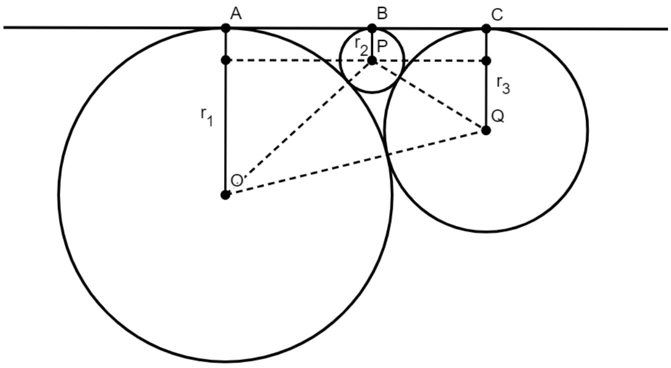
2 BMO2 2016

2.1 Problem 1

Circles of radius r_1, r_2 and r_3 touch each other externally, and they touch a common tangent at points A, B and C respectively, where B lies between A and C. Prove that

$$16(r_1 + r_2 + r_3) \geq 9(AB + BC + CA)$$

Solution. The fact we are being asked to prove is not necessarily completely obvious, and it is difficult to see which direction to head in after having examined the setup. But, we will deduce what we can from the setup and potentially working backwards will be helpful in seeing how the result is derived.



As can be seen from the diagram, the centres of each of the circles are labelled O, P and Q. Clearly, $OP = r_1 + r_2$, $PQ = r_2 + r_3$ and $OQ = r_1 + r_3$, since all the circles meet tangentially. We will now aim to obtain expressions for AB, BC and CA in terms of r_1, r_2 and r_3.

By Pythagoras', $AB^2 = OP^2 - (r_1 - r_2)^2 = (r_1 + r_2)^2 - (r_1 - r_2)^2 = 4r_1 r_2 \implies AB = 2\sqrt{r_1 r_2}$. Similarly, $BC = 2\sqrt{r_2 r_3}$ and $CA = 2\sqrt{r_1 r_3}$. So, the statement required to prove reduces to

$$16\,(r_1 + r_2 + r_3) \geq 9\,(2\sqrt{r_1 r_2} + 2\sqrt{r_2 r_3} + 2\sqrt{r_1 r_3})$$
$$8\,(r_1 + r_2 + r_3) \geq 9\,(\sqrt{r_1 r_2} + \sqrt{r_2 r_3} + \sqrt{r_1 r_3})$$

In order to avoid dealing with square roots, it would be helpful to make a substitution at this point. Let $x = \sqrt{r_1 r_2}, y = \sqrt{r_2 r_3}, z = \sqrt{r_1 r_3}$.

$$\implies r_1 = \frac{xz}{y},\ r_2 = \frac{xy}{z},\ r_3 = \frac{yz}{x}$$

We also know from the setup $AB + BC = AC \implies 2x + 2y = 2x \implies x + y = z$. Using these constraints, we have turned our problem into an algebraic inequality that we must prove. So, we must show that

$$16\left(\frac{xz}{y} + \frac{xy}{z} + \frac{yz}{x}\right) \geq 18(x + y + z)$$
$$8(x^2z^2 + x^2y^2 + y^2z^2) \geq 9(xyz)(x + y + z)$$

But since $x + y = z$, we can reduce the inequality down two just two variables x and y. So, we get

$$4\left(x^2(x+y)^2 + y^2(x+y)^2 + x^2y^2\right) \geq 9xy(x+y)^2$$

While it may not be the most elegant thing, what we can do now is fully expand and simplify the terms on the right hand side and the left hand side. Doing so yields

$$4x^4 + 4y^4 + 12x^2y^2 + 8x^3y + 8xy^3 \geq 9x^3y + 9xy^3 + 18x^2y^2$$

So, the expression finally reduces and we are required to prove:

$$4x^4 + 4y^4 \geq x^3y + xy^3 + 6x^2y^2 \tag{4}$$

But (4) is not too difficult to prove true. By AM-GM

$$\frac{x^4 + y^4}{2} \geq \sqrt{x^4y^4} \implies x^4 + y^4 \geq 2x^2y^2 \tag{5}$$

Also, by rearrangement inequality

$$x^4 + y^4 \geq x^3y + xy^3 \tag{6}$$

Now doing $3 \times (5) + (6)$

$$\begin{aligned} 3x^4 + 3y^4 + x^4 + y^4 &\geq 6x^2 + y^2 + x^3y + xy^3 \\ 4x^4 + 4y^4 &\geq x^3y + xy^3 + 6x^2y^2 \end{aligned}$$

which demonstrates that (4) is true and hence the logic follows backwards, thus it is true that

$$16(r_1 + r_2 + r_3) \geq 9(AB + BC + CA) \quad .$$

□

2.2 Problem 4

Suppose that p is a prime number and that there are different positive integers u and v such that p^2 is the mean of u^2 and v^2. Prove that $2p - u - v$ is a square or twice a square.

Solution. From the question we have,

$$p^2 = \frac{u^2 + v^2}{2} \implies 2p^2 = u^2 + v^2$$

Now, we want to somehow 'manufacture' this $2p-u-v$ to begin considering how it could be a perfect square. Multiplying both sides by 2 helps us with this.

$$4p^2 = 2u^2 + v^2 = \left(u^2 - v^2\right) + \left(u^2 + v^2\right)$$

$$\therefore \; 4p^2 - (u+v)^2 = (u-v)^2 \implies \underline{(2p - u - v)(2p + u + v) = (u - v)^2}$$

We only need to examine $2p - u - v$ from the above. The expression above alludes to a somewhat 'obvious' result: if we have two coprime numbers m, n (*i.e.* $\gcd(m, n) = 1$) and if mn is a perfect square, then both m and n are also square numbers. This way, it is handy to consider *common divisors* of $2p - u - v$ and $2p + u + v$.

Let q be a common prime factor of both $2p - u - v$ and $2p + u + v$. If q divides both these, then it must also divide their sum $((2p - u - v) + (2p + u + v) = 4p)$ and their difference $((2p + u + v) - (2p - u - v) = 2(u + v))$. So,

$$q|4p \;\;,\quad q|2(u+v)$$

Let's first suppose the case that q is odd. Since $q|4p$, it must be that $q|p$, but further since p is prime, $q = p$. As a result, $p|2(u + v)$ which means that $p|(u + v)$.

But, right from the beginning, we established that $p|(u^2 + v^2)$ (as $2p^2 = u^2 + v^2$). Hence p also divides the following difference,

$$p|(u+v)^2 - (u^2 + v^2) = 2uv \implies p|uv$$

However, if p for example divides u then it also divides v as $p|u + v$. So, both u and v are divisible by p and we can write that $u = Ap$ and $v = Bp$ for some factors A and B.

$$\therefore \; 2p^2 = (Ap)^2 + (Bp^2) = p^2 \left(A^2 + B^2\right)$$

Though, because A and B are integer factors it should be clear that $A = B = 1$, and thus $p = u = v$. But we can't have this as $u \neq v$ by the question.

So, we have established that the common prime factor q cannot be odd. Hence it must be 2, and the greatest common divisor of $2p - u - v$ and $2p + u + v$ must be some power of 2, say 2^n.

Going back to the equation we established,

$$(2p - u - v)(2p + u + v) = (u - v)^2$$

we must have that $2p - u - v = 2^n x^2$ and $2p + u + v = 2^n y^2$ for some numbers x and y. This is because $\gcd(2p - u - v, 2p + u + v) = 2^n$ and the right hand side is a perfect square which means the left hand side must also be.

Since $2p - u - v = 2^n x^2$, if n is even then it is a square, and if n is odd then it is twice a square. □

3 BMO2 2020

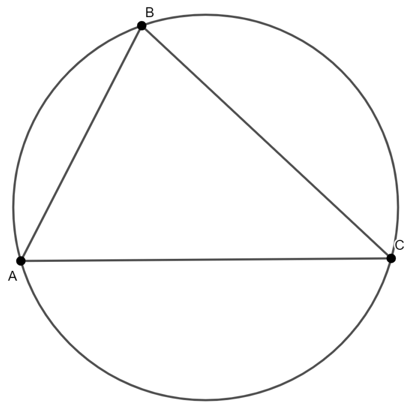
3.1 Problem 2

Describe all collections S of at least four points in the plane such that no three points are collinear and such that every triangle formed by three points in S has the same circumradius.

(*The circumradius of a triangle is the radius of the circle passing through all three of its vertices.*)

Solution. This problem involves us investigating properties of some points in the plane. Since we must have at least 4 points, we can start with arbitrarily picking any 3 non-collinear points which form some arbitrary triangle.

We can do this because our three points will initially define a circumcircle (and so a circumradius) uniquely, and then after we can investigate what we do with the next fourth point and then the next point and so on. So, we start with the following.

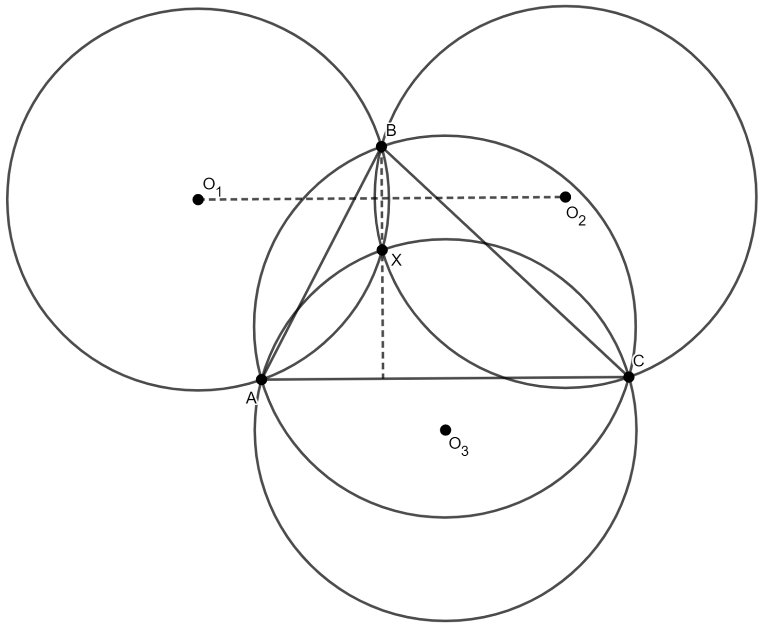


Now, we want to find a fourth point, say X, such that the circumcircle formed by AXB, BXC and CXA all have the same circumradius.

Of course, if our point X lies on the circumcircle of ABC, then that is good enough, since then AXB, BXC and CXA all have exactly the same circumcircle so the same circumradius. Similarly, we can do this for points after that, by simply making all the points *concyclic*.

However, we must think - is this the only way to create circumcircles with the same radius? We can play around with this a little bit. Essentially, what we want now is that our fourth point X does not lie on the circumcircle of ABC, but it still satisfies the condition that AXB, BXC and CXA have the same circumradius.

This will give us the following setup.



Notice that we must have the three circumcircles with the same radius intersecting at the point X. Now, we want to investigate the properties of this point X. A helpful way to proceed is by joining the centres of two of the circles to form a line say O_1O_2.[2]

Now, what we have in our diagram are several rhombuses at play. Due to the circles being constrained to have the same radii, BO_1XO_2 is a rhombus, so it's diagonals are perpendicular, hence $O_1O_2 \perp BX$.

We will bring rhombuses AO_3XO_1, CO_2XO_3 and BO_1XO_2 into play. Let's call $\angle AO_3O_1 = \alpha$ and $\angle CO_2O_3 = \beta$. By diagonals in a rhombus bisecting the angles we see that $\angle AO_3C = 2(\alpha + \beta) \implies \angle O_1O_3O_2 = \alpha + \beta$ (draw the rhombuses if this is unclear).

But, $O_1X = O_2X = O_3X$, so $\triangle O_1O_3O_2$ has center X! Hence, by angle at centre is twice angle at circumference, $\angle O_1O_3O_2 = 2\angle O_1XO_2$, so $\angle O_1XO_2 = \angle AO_3C$. Since opposite angles in a rhombus are equal, we get $\underline{\angle AO_3C = \angle O_2BO_1}$.

Therefore, $\triangle BO_1O_2$ is congruent to $\triangle AO_3C$ by SAS. Hence, $O_1O_2 = AC$ and $AO_1 = CO_2$ (by the same radii), so ACO_2O_1 is a parallelogram and O_1O_2 is parallel to AC. It then follows that $BX \perp AC$ as well.

We can follow this argument round using lines O_1O_3 and O_2O_3, and we will similarly find that $AX \perp BC$ and $CX \perp AB$. So X is the *orthocentre* of $\triangle ABC$!

So, given three points in the plane A, B, C, the next point can either lie on the circumcircle of ABC or on the orthocentre of ABC. Since there is only one unique orthocentre, once a point is on the orthocentre, the other points must fill up on the circumcircle of ABC. $\square$

[2]It doesn't really matter which two centres we choose; the argument remains the same due to the symmetry of the problem.

4 BMO2 2021

4.1 Problem 1

A positive integer n is called *good* if there is a set of divisors of n whose members sum to n and include 1. Prove that every positive integer has a multiple which is good.

Solution. We have $n \in \mathbb{Z}^+$ is good if there exists a set $\mathcal{S}$ such that:

$$\mathcal{S} = \{1, a_1, a_2, \dots, a_k\}$$

where all members of $\mathcal{S}$ are divisible by n *and* $1 + a_1 + a_2 + \cdots + a_k = n$.

Clearly, n can never be good and prime, since the divisors of n are 1 and n, and $1 \neq n$ or $1 + n \neq n$. Trying small examples, we can immediately see that $n = 6$ is good, since when $n = 6$, $\mathcal{S} = \{1, 2, 3\}$. It is tempting to suppose that all multiples of 6 are good, since both 12 and 18 are. However, 66 is not good, and in fact any number of the form $6p$, where $p \in \mathbb{P}$ and $p \geq 11$, is never good.

Naturally, it would make sense to try the next best case of multiples of 12, however again we encounter a similar problem. $12 \times 29 = 348$ is not good, and in fact any number of the form $12p$, where $p \in \mathbb{P}$ and $p \geq 29$, is never good.

More investigating is required here, and perhaps it is not sensible to go down the route of simple 'multiples'.

$$\begin{aligned} n &= 6 & \mathcal{S} &= \{1, 2, 3\} \\ n &= 12 & \mathcal{S} &= \{1, 2, 3, 6\} \\ n &= 24 & \mathcal{S} &= \{1, 2, 3, 6, 12\} \end{aligned}$$

After looking at some examples of good numbers, we can make a key observation:

If k is a good number, then $2k$ is also a good number. Further, if k is a good number, then $2^x k$ is also a good number for all positive integers x.

To prove this, let's suppose we have our number k which is good. Thus, there must exist a set $\mathcal{S} = \{1, a_1, a_2, \dots, a_r\}$, with $1, a_1, a_2, \dots, a_r$ being divisors of k and $k = 1 + a_1 + a_2 + \cdots + a_r$. Now, if we add the number k itself to $\mathcal{S}$, then the sum of members in $\mathcal{S}$ becomes

$$1 + a_1 + a_2 + \cdots + a_r + k = k + k = 2k$$

and since k is divisible by all of $1, a_1, a_2, \dots, a_r$, $2k$ is also divisible by them as well as k, so $2k$ is also good. The logic follows repetitively, so multiplying by 2 creates more good numbers, and hence $2^x k$ is always good too.

In the examples illustrated above, we have considered examples with $n = 2^x \times 3$, however we can generalise this further to $n = 2^x \times p$, where p is **any odd** integer.

I CONJECTURE that all numbers of the form $n = 2^x \times p$ are good. Clearly, however, there needs to be some restrictions on the value of x since $2 \times 7 = 14$ is not good, but $2^2 \times 7 = 28$ is good.

Consider $n = 2^x \times p$. We take p as being odd. It is a common fact that every number can be expressed as the sum of distinct powers of two,[3] and more specifically if the integer is

[3] This is equivalent to saying that every base 10 number can be written in binary.

odd then when it is expressed as a sum of distinct powers of two, one of the powers is $2^0 = 1$. So, we can express p as

$$p = 2^0 + 2^{a_1} + 2^{a_2} + \cdots + 2^{a_k} = 1 + 2^{a_1} + 2^{a_2} + \cdots + 2^{a_k}$$

Now, we can write $n = 2^x \times p$ as

$$n = 2^x p - p + p = (2^x - 1)\, p + p$$

Recall the common factorisation $(2^x - 1) = (2 - 1)\,(2^{x-1} + 2^{x-2} + \cdots + 1) = 1 + 2 + \cdots + 2^{x-2} + 2^{x-1}$.

$$\begin{aligned} \therefore\ \ n &= \left(1 + 2 + \cdots + 2^{x-2} + 2^{x-1}\right) p + \left(1 + 2^{a_1} + 2^{a_2} + \cdots + 2^{a_k}\right) \\ &= \boxed{1 + 2^{a_1} + 2^{a_2} + \cdots + 2^{a_k} + p + 2p + \cdots + 2^{x-2}p + 2^{x-1}p} \end{aligned}$$

We can clearly see that the above expression for n is a sum of various factors of n. Thus, $n = 2^x \times p$ is good. However, we must have x being large for safety (which isn't a problem at all), so that we can have the full sum of factors hold.

Hence, we have proved that numbers of the form $n = 2^x \times p$ for any large, positive integer x and positive, odd integer p are good.

Thus, n is obviously a multiple of all odd integers, and it is also a multiple of all even integers by suitably choosing a p and large enough x. □

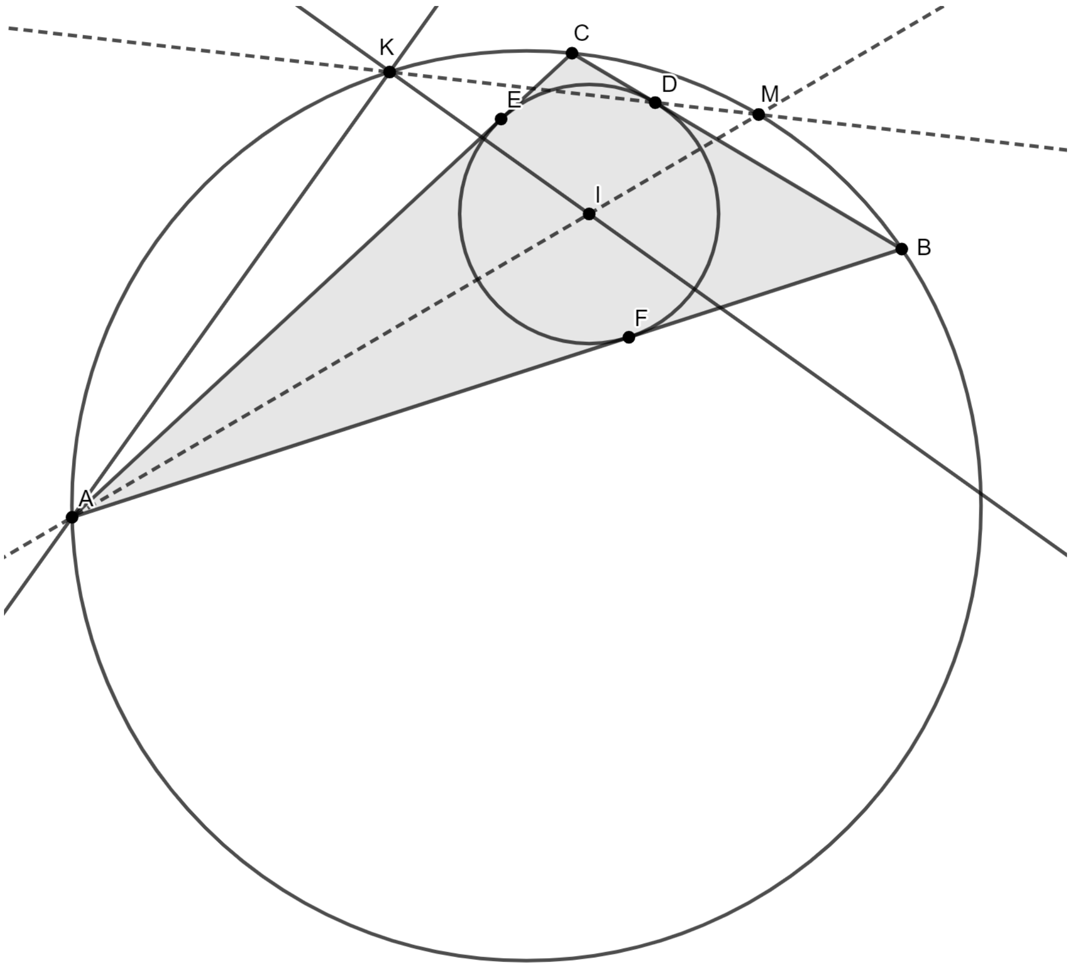
4.2 Problem 3

Let ABC be a triangle with $AB > AC$. Its circumcircle is Γ and its incentre is I. Let D be the contact point of the incircle of ABC with BC.

Let K be the point on Γ such that $\angle AKI$ is a right angle.

Prove that AI and KD meet on Γ.

Solution. Let the lines AI and KD meet at a point M, and let points E and F lie on lines AC and AB respectively such that IE is perpendicular to AC and IF is perpendicular to AB.



We start by proving that $\triangle KEC$ is similar to $\triangle KFB$. Since the point E lies on AC and point F lies on AB, $\angle ECK = \angle FBK$ by angles in the same segment (chord AK).

Now, note that points A, F, I, E, K are concyclic with diameter AI. This is because $AFIE$ is cyclic as $\angle AFI + \angle IEA = 90° + 90° = 180°$, and $\angle IKA = 90° = \angle AIE$ so K lies on this circle too by converse of angles in the same segment.

Thus, by angles in same segment (chord AK), $\angle AEK = \angle AFK \implies \angle KEC = \angle KFB$.

$$\therefore\ \triangle KEC \sim \triangle KFB \text{ (by AA)}$$

So we can deduce the following relation.

$$\frac{KC}{KB} = \frac{CE}{BF}$$

But, $CE = CD$ and $BF = BD$, since tangents from the same point are of equal length.[4]

$$\therefore \frac{KC}{KB} = \frac{CD}{BD} \tag{7}$$

The relation in (7) implies that line KDM is the internal bisector of $\angle CKB$, by converse of the angle bisector theorem.

Now, $\angle CKB = \angle CAB$ by angles in the same segment (chord BC). Let $\angle CKM = \alpha$, so $\angle CKB = 2\alpha = \angle CAB$.

But, line AM passes through the point I (the incentre) so line AIM is the internal angle bisector of $\angle CAB$. This is because $\triangle AIE \cong \triangle AFI$ by SAS. So, $\angle MAB = \alpha$.

Hence, since $\angle MKB = \alpha = \angle CKM$, then by converse of angles in the same segment, point M lies on the circumcircle Γ of ABC. □

[4] These lines are tangents to the incircle because the radii points of contact are at 90°, and we apply converse of 'radius perpendicular to tangent'.