Introduction to Differential Geometry

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Differential geometry is concerned with studying the geometry of '*smooth surfaces*'. In this document, we provide an introduction to the classical theory of differential geometry, where we discuss geometric features of curves and surface in (mostly) three-dimensional Euclidean spaces. Most prominently, differential geometry is the language of the famous theory of general relativity, but it has also seen applications in computer graphics, computer vision and, recently, machine learning.

In this course, we assume familiarity with the foundations of multivariable calculus, and basic linear algebra and analysis. This document constitutes my notes taken from lectures by *Prof. Steven Sivek* at *Imperial College London.*

To be completed. Content will cover up to and including the complete Gauss-Bonnet theorem.

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§1 Curves

In this document, we focus mostly on **regular curves** in $\mathbb{R}^2$ and $\mathbb{R}^3$, and we assume **smoothness** for simplicity (although it would be sufficient to assume a finite number of continuous derivative *e.g.* C^4).

Another note for the reader: while I have tried to provide as much intuition throughout this document, geometry (at least in $\mathbb{R}^2$ and $\mathbb{R}^3$) is inherently visual, and as such I would encourage looking up how the curves and surfaces discussed here actually look like.

§1.1 Curves in Euclidean space

We begin by studying some basic properties of curves in **Euclidean space**.

Definition 1.1 (Regular Parametrized Curve) — A **parametrized curve** is a *smooth* map $\phi : [a, b] \to \mathbb{R}^n$, and it is called **regular** if $\phi'(t) \neq 0$ for all t.

You should think of a parametrized curve as describing the motion of a particle over time.

The parameter t plays the role of time, and the derivative $\phi'(t)$ is the *velocity vector* of the particle at time t. The regularity condition ensures that the particle is always moving and never comes to rest.

Example 1.2 An example of a regular curve in $\mathbb{R}^2$ is a circle parametrized by $\phi_1 : [0, 2\pi] \to \mathbb{R}^2$, $\phi_1(t) = (\cos t, \sin t)$, is a regular curve.

An example of a regular curve in $\mathbb{R}^3$ is given by $\phi_2 : [0, 1] \to \mathbb{R}^3$, $\phi_2(t) = (t^2,\, 1-t,\, t^4)$.

Example 1.3 The following curves are *not* regular:

$$\phi_3(t) = (t^2, t^3), \qquad \phi_4(t) = (t^2 + t, |t|).$$

The curve ϕ_3 fails to be regular since $\phi_3'(0) = 0$, while ϕ_4 is not differentiable at $t = 0$.

Regularity excludes degenerate cases such as constant curves $\phi(t) = (1, 2, 3)$, and ensures that at every point of the curve there is a well-defined **tangent line**. At the point $\phi(t_0)$, the tangent line is $L = \{\phi(t_0) + s\,\phi'(t_0) \mid s \in \mathbb{R}\}$.

If you imagine travelling along the curve, the tangent vector $\phi'(t_0)$ points in the direction you would continue moving at time t_0.

In many situations, we are interested not in a *specific parametrization*, but in the geometric *image* of the curve. For example, $\tilde{\phi}_1(t) = (\cos(2t), \sin(2t))$ for $t \in [0, \pi]$, traces out the same circle as ϕ_1, but at a different speed.

Definition 1.4 (Reparametrization) — Let $\phi : [a, b] \to \mathbb{R}^n$ be a regular curve, and let $f : [c, d] \xrightarrow{\sim} [a, b]$ be a smooth function with $f'(t) \neq 0$ for all t. The curve

$$\phi \circ f : [c, d] \to \mathbb{R}^n, \quad t \mapsto \phi(f(t)),$$

is called a **reparametrization** of ϕ.

Remark 1.5 A reparametrization of a regular curve is again regular, since $(\phi \circ f)'(t) = \phi'(f(t))\, f'(t)$, which is nonzero because both factors are nonzero.

Reparametrization corresponds to changing the *speed* at which we traverse the curve, possibly reversing direction, but not changing the geometric path itself.

We now focus on properties of curves that do not depend on the chosen parametrization. To do this, we use the standard inner product on $\mathbb{R}^n$, $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$, and the induced norm $|x| = \sqrt{\langle x, x\rangle}$.

Definition 1.6 (Length of a curve) — The **length** of a curve $\phi : [a, b] \to \mathbb{R}^n$ is

$$L(\phi) = \int_a^b |\phi'(t)|\, dt.$$

Remark 1.7 Since we established that the quantity $|\phi'(t)|$ is the speed of the particle at time t, the length of the curve is therefore the total distance travelled.

Proposition 1.8 The length of a regular curve in $\mathbb{R}^n$ is invariant under reparametrization.

Proof. Let $\psi = \phi \circ f$ be a reparametrization of ϕ, with $f'(t) > 0$. Then $|\psi'(t)| = |\phi'(f(t))|\, f'(t)$. Hence,

$$L(\psi) = \int_c^d |\phi'(f(t))|\, f'(t)\, dt = \int_a^b |\phi'(s)|\, ds = L(\phi),$$

where we used the substitution $s = f(t)$. □

Since length is invariant under reparametrization, it is natural to look for a particularly *convenient* parametrization.

Definition 1.9 (Arclength parametrization) — A curve $\phi : [a, b] \to \mathbb{R}^n$ is **parametrized by arclength** if $|\phi'(t)| = 1$ for all t.

In this case, $L(\phi) = \int_a^b dt = b - a$, and equivalently, $L(\phi|_{[a,t]}) = t - a$.

Remark 1.10 Parametrizing by arclength means travelling along the curve at unit speed. The parameter t literally measures the distance travelled along the curve.

Proposition 1.11 Every regular curve $\phi : [a, b] \to \mathbb{R}^n$ admits an arclength parametrization.

Proof. Define the arclength function $\ell(t) = \int_a^t |\phi'(s)|\, ds$. Since $\ell'(t) = |\phi'(t)| > 0$, the function ℓ is strictly increasing and invertible.

Let $f = \ell^{-1}$, and define $\psi = \phi \circ f$. Differentiating $\ell(f(t)) = t$ gives $|\phi'(f(t))|\, f'(t) = 1$, so $|\psi'(t)| = 1$, as required. □

§1.2 Curvature

Definition 1.12 (Curvature for arclength parametrized curves) — Let $\phi : [a, b] \to \mathbb{R}^n$ be a curve parametrized by arclength, i.e. $|\phi'(t)| = 1$ for all t. Define the **curvature vector** and **curvature** by $\vec{\kappa}(t) = \phi''(t)$, $\kappa(t) = |\phi''(t)|$, respectively.

The curvature vector is independent from arclength reparametrization. If $\psi(t) = \phi(f(t))$ is another arclength parametrization of the same curve, then $1 = |\psi'(t)| = |\phi'(f(t))|\,|f'(t)| = 1 \cdot |f'(t)|$, so $|f'(t)| = 1$, hence $f(t) = \pm t + C$. Differentiating twice gives $\psi''(t) = \phi''(f(t))$, so the curvature vector (and therefore κ) is the same geometric object along the curve, independent of which arclength parametrization we chose.

Proposition 1.13 (Zero curvature $\iff$ straight line) Let $\phi : [a, b] \to \mathbb{R}^n$ be arclength parametrized. Then $\kappa(t) = 0$ for all t if and only if ϕ traces a straight line.

Proof. Since $\kappa(t) = |\phi''(t)|$, we have $\kappa(t) = 0 \iff \phi''(t) = 0$. If $\phi''(t) = 0$ for all t, then $\phi'(t)$ is constant, say $\phi'(t) = v$, and hence $\phi(t) = \phi(a) + v(t - a)$, a straight line.

Conversely, any straight line parametrized by arclength has constant velocity and thus $\phi''(t) = 0$, so $\kappa(t) = 0$. □

Proposition 1.14 (Curvature vector is orthogonal to the tangent) If ϕ is arclength parametrized, then $\phi''(t) \perp \phi'(t)$ for all t.

Proof. Arclength parametrization means $|\phi'(t)|^2 = \langle \phi'(t), \phi'(t) \rangle = 1$ is constant. Differentiate:

$$0 = \frac{d}{dt}\langle \phi'(t), \phi'(t) \rangle = 2\langle \phi''(t), \phi'(t) \rangle,$$

so $\langle \phi''(t), \phi'(t) \rangle = 0$, i.e. $\phi''(t) \perp \phi'(t)$. □

Example 1.15 (Curvature of a circle of radius R) Consider the circle of radius $R > 0$ in $\mathbb{R}^2$. A standard parametrization is $\phi(t) = (R\cos t, R\sin t)$, $t \in [0, 2\pi]$.

Here $|\phi'(t)| = R$, so this is not arclength parametrized. Since the arclength from 0 to t equals Rt, we reparametrize by $t \mapsto t/R$ and obtain the arclength parametrization

$$\psi(t) = \Big(R\cos\frac{t}{R},\ R\sin\frac{t}{R}\Big), \qquad t \in [0, 2\pi R].$$

Then

$$\psi''(t) = \Big(-\frac{1}{R}\cos\frac{t}{R}, -\frac{1}{R}\sin\frac{t}{R}\Big) = -\frac{1}{R^2}\psi(t),$$

so

$$\kappa(t) = |\psi''(t)| = \frac{1}{R}.$$

Thus a circle of radius R has constant curvature $1/R$.

§1.3 The Frenet frame

Let $\phi : [a,b] \to \mathbb{R}^3$ be arclength parametrized, so $T(t) := \phi'(t)$ is a unit tangent vector. When $\phi''(t) \neq 0$ (equivalently $\kappa(t) \neq 0$), the curvature vector provides a preferred normal direction, and we can build an orthonormal moving frame along the curve.

Definition 1.16 (Frenet frame) — Let $\phi : [a,b] \to \mathbb{R}^3$ be parametrized by arclength and assume $\phi''(t) \neq 0$. Define:

- the **unit tangent vector** $T(t) = \phi'(t)$ (which has unit length);
- the **principal normal vector** $N(t) = \frac{T'(t)}{|T'(t)|}$ (if $T'(t) \neq 0$);
- the **binormal vector** $B(t) = T(t) \times N(t)$ (if $T'(t) \neq 0$).

Then $(T(t), N(t), B(t))$ is an orthonormal, positively oriented basis of $\mathbb{R}^3$ at $\phi(t)$, called the **Frenet frame**.

Because $T'(t)$ points in the normal direction and has magnitude $\kappa(t)$, we immediately get

$$T'(t) = \kappa(t)N(t).$$

Next, $B(t)$ is unit length, so $B'(t) \perp B(t)$; and since $B = T \times N$, differentiating gives $B'(t) = T(t) \times N'(t)$, so $B'(t) \perp T(t)$ as well (a cross product is orthogonal to its factors). Hence $B'(t)$ is orthogonal to both $B(t)$ and $T(t)$, so it must be parallel to $N(t)$. We therefore define the **torsion** $\tau(t)$ by

$$B'(t) = -\tau(t)N(t).$$

Finally, using $N = B \times T$ and differentiating yields

$$N'(t) = \tau(t)B(t) - \kappa(t)T(t).$$

Proposition 1.17 (Frenet formulas) Assume ϕ is arclength parametrized in $\mathbb{R}^3$ with $\phi''(t) \neq 0$. Then

$$\begin{aligned} T'(t) &= \kappa(t)N(t), \\ N'(t) &= -\kappa(t)T(t) + \tau(t)B(t), \\ B'(t) &= -\tau(t)N(t). \end{aligned}$$

Equivalently,

$$\frac{d}{dt}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Remark 1.18 Curvature measures how strongly the curve bends away from its tangent direction.

Torsion measures how the curve twists out of the plane spanned by T and N; morally, it measures how non-planar the curve is.

Proposition 1.19 (Planarity $\iff$ zero torsion) Let $\phi : [a, b] \to \mathbb{R}^3$ be arclength parametrized with $\phi''(t) \neq 0$ for all t. Then ϕ lies in a plane if and only if $\tau(t) = 0$ for all t.

Proof. If $\tau(t) = 0$, then $B'(t) = -\tau(t)N(t) = 0$, so $B(t) = \vec{c}$ is constant. Then

$$\frac{d}{dt}\langle \phi(t), \vec{c}\rangle = \langle \phi'(t), \vec{c}\rangle = \langle T(t), B(t)\rangle = 0,$$

so $\langle \phi(t), \vec{c}\rangle = d$ is constant and $\phi(t)$ lies in the plane $\{\vec{x} : \langle \vec{x}, \vec{c}\rangle = d\}$.

Conversely, if $\phi(t)$ lies in a plane $\langle \vec{x}, \vec{c}\rangle = d$ with $\|\vec{c}\| = 1$, then $\langle \phi(t), \vec{c}\rangle = d$ implies $\langle T(t), \vec{c}\rangle = 0$, and differentiating again gives

$$0 = \langle T'(t), \vec{c}\rangle = \langle \kappa(t)N(t), \vec{c}\rangle.$$

Since $\kappa(t) \neq 0$, we get $\langle N(t), \vec{c}\rangle = 0$. Thus $\vec{c}$ is orthogonal to both T and N, so $B = \pm\vec{c}$ is constant, hence $B'(t) = 0$ and $0 = B'(t) = -\tau(t)N(t)$ forces $\tau(t) = 0$. □

Example 1.20 (Helix) Define $\phi(t) = \left(\cos\frac{t}{\sqrt{2}},\ \sin\frac{t}{\sqrt{2}},\ \frac{t}{\sqrt{2}}\right)$ for $t \in \mathbb{R}$. The shape of this curve is similar to that of a single strand of DNA molecule. It is worth looking up what this looks like for intuition, and it will become clear that it does not lie in any one plane.

Then $\phi'(t) = \left(-\frac{1}{\sqrt{2}}\sin\frac{t}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\cos\frac{t}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)$, so $|\phi'(t)| = 1$ and the curve is arclength parametrized.

Differentiating again, $T'(t) = \phi''(t) = \left(-\frac{1}{2}\cos\frac{t}{\sqrt{2}},\ -\frac{1}{2}\sin\frac{t}{\sqrt{2}},\ 0\right)$, hence

$$\kappa(t) = |T'(t)| = \frac{1}{2}, \quad N(t) = \frac{T'(t)}{|T'(t)|} = \left(-\cos\frac{t}{\sqrt{2}},\ -\sin\frac{t}{\sqrt{2}},\ 0\right).$$

Compute $B = T \times N$, $B(t) = \left(\frac{1}{\sqrt{2}}\sin\frac{t}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\cos\frac{t}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)$, and then

$$B'(t) = \left(\frac{1}{2}\cos\frac{t}{\sqrt{2}},\ \frac{1}{2}\sin\frac{t}{\sqrt{2}},\ 0\right) = -\frac{1}{2}N(t),$$

so by $B' = -\tau N$ we obtain $\tau(t) = \frac{1}{2}$. Thus, the helix is not planar.

So far, we have seen that a regular space curve becomes especially easy to understand once it is *parametrized by arclength*. Intuitively, we think of t as "time" and $\phi(t)$ as the position of a particle.

The curvature measures how quickly the unit tangent T turns as we move along the curve, and the torsion measures how the curve twists out of its osculating plane; in particular, $\tau \equiv 0$ means the curve stays in a plane, as we established.

Our next result states that in fact κ and τ determine the curve ϕ, essentially uniquely.

Theorem 1.21 (Fundamental theorem of the local theory of curves) Let $\kappa, \tau : [a, b] \to \mathbb{R}$ be smooth functions with $\kappa(t) > 0$ for all $t \in [a, b]$.

1. *Existence.* There exists a regular curve $\phi : [a, b] \to \mathbb{R}^3$, parametrized by arclength, whose curvature and torsion are exactly $\kappa(t)$ and $\tau(t)$.
2. *Uniqueness up to rigid motion.* If $\psi : [a, b] \to \mathbb{R}^3$ is another arclength-parametrized curve with the same curvature and torsion, then ψ differs from ϕ by a rigid motion:
$$\psi(t) = g \circ \phi(t) + c$$
for some fixed rotation $g \in SO(3)$ and translation vector $c \in \mathbb{R}^3$.

Here's some intuition about this theorem: curvature $\kappa(t)$ tells you 'how hard you are steering' as you travel along the curve; torsion $\tau(t)$ tells you 'how much the steering wheel is twisting the plane of motion.'

The theorem states: if you prescribe these two steering instructions as functions of time (with $\kappa > 0$), then there is a curve that follows them, and any two such curves are the same shape in space, just rotated and translated.

Example 1.22 (Constant curvature, zero torsion $\Rightarrow$ circle) Let $\phi : [a, b] \to \mathbb{R}^3$ have torsion $\tau(t) = 0$ and constant curvature $\kappa(t) = c > 0$. Since $\tau \equiv 0$, the curve is planar (by the planarity criterion $\tau \equiv 0$). Now consider the explicit plane curve

$$\psi(t) = \left(\frac{1}{c} \cos(ct),\ \frac{1}{c} \sin(ct),\ 0 \right),$$

which is parametrized by arclength and has constant curvature c (exactly as for a circle of radius $1/c$). It also lies in the plane $z = 0$, so $\tau \equiv 0$. Therefore, by theorem 1.21, ϕ must be obtained from ψ by a rigid motion. Hence ϕ is (a piece of) a circle of radius $1/c$ in some plane in $\mathbb{R}^3$.

§1.4 Plane curves

Let $\phi : [a, b] \to \mathbb{R}^2$ be a regular curve $\phi(t) = (x(t), y(t))$. Then the velocity $\phi'(t) = (x'(t), y'(t))$ is nonzero and points along the tangent direction. A convenient *unit tangent* is $T(t) = \frac{\phi'(t)}{|\phi'(t)|}$.

A natural choice of unit normal (rotating T by $+\pi/2$) is $N(t) = \frac{1}{|\phi'(t)|}(-y'(t), x'(t))$.

This choice makes (T, N) a positively oriented orthonormal basis of $\mathbb{R}^2$:

$$\det \begin{pmatrix} | & | \\ T & N \\ | & | \end{pmatrix} = +1.$$

If ϕ is parametrized by arclength (so $|\phi'(t)| = 1$), then $\phi''(t)$ is orthogonal to $\phi'(t)$ and $\phi''(t) = \kappa(t)N(t)$. Taking the dot product with $N(t)$ gives $\kappa(t) = \langle \phi''(t), N(t) \rangle$. Writing this in coordinates with $|\phi'(t)| = 1$ yields the familiar determinant-type formula:

$$\kappa(t) = x'(t)y''(t) - y'(t)x''(t).$$

Even if ϕ is *not* parametrized by arclength, curvature should be a geometric quantity that does not depend on how fast we traverse the curve. This is expressed in the following proposition.

Proposition 1.23 Let $\phi : [a,b] \to \mathbb{R}^2$ be a regular curve (not necessarily arclength-parametrized). Then its curvature is

$$\kappa(t) = \frac{\langle \phi''(t), N(t) \rangle}{|\phi'(t)|^2} = \frac{x'(t)y''(t) - y'(t)x''(t)}{\left(x'(t)^2 + y'(t)^2\right)^{3/2}}.$$

Proof. Let ψ be an arclength reparametrization of ϕ, say $\psi = \phi \circ f$. Then

$$\psi'(t) = \phi'(f(t))\, f'(t), \quad \psi''(t) = \phi''(f(t))\,(f'(t))^2 + \phi'(f(t))\, f''(t).$$

For plane curves, curvature in arclength parameter satisfies

$$\kappa_\psi(t) = \langle \psi''(t), N_\psi(t) \rangle.$$

Since $N_\psi(t)$ is orthogonal to $\psi'(t)$, it is also orthogonal to $\phi'(f(t))$, so the $\phi'(f(t))\, f''(t)$ term drops out:

$$\kappa_\psi(t) = \langle \phi''(f(t))\,(f'(t))^2,\, N_\psi(t) \rangle = (f'(t))^2 \langle \phi''(f(t)), N_\psi(t) \rangle.$$

Also $N_\psi(t) = N_\phi(f(t))$ (same geometric normal at the same point), and for arclength reparametrization one has

$$|\psi'(t)| = 1 \quad \Rightarrow \quad |\phi'(f(t))|\, f'(t) = 1 \quad \Rightarrow \quad f'(t) = \frac{1}{|\phi'(f(t))|}.$$

Substituting gives

$$\kappa_\phi(f(t)) = \kappa_\psi(t) = \frac{\langle \phi''(f(t)), N_\phi(f(t)) \rangle}{|\phi'(f(t))|^2}.$$

Rename $s = f(t)$ to obtain $\kappa(s) = \langle \phi''(s), N(s) \rangle / |\phi'(s)|^2$. ☐

Example 1.24 (Curvature of a graph) Let $\phi(t) = (t, f(t))$ be the graph of a smooth function f. Then

$$\phi'(t) = (1, f'(t)), \quad \phi''(t) = (0, f''(t)), \quad |\phi'(t)| = \sqrt{1 + (f'(t))^2}.$$

A compatible unit normal is

$$N(t) = \frac{1}{\sqrt{1 + (f'(t))^2}} (-f'(t), 1).$$

Therefore

$$\kappa(t) = \frac{\langle \phi''(t), N(t) \rangle}{|\phi'(t)|^2} = \frac{f''(t)/\sqrt{1 + (f'(t))^2}}{1 + (f'(t))^2} = \frac{f''(t)}{\left(1 + (f'(t))^2\right)^{3/2}}.$$

§1.4.1 Winding number, turning number, and total curvature

For a **closed plane curve**, curvature connects *geometry* (how much the curve bends) with *topology* (how many times it winds/turns).

Definition 1.25 (Closed curve) — A smooth curve $\phi : [a,b] \to \mathbb{R}^2$ is *closed* if $\phi(a) = \phi(b)$ and $\phi^{(k)}(a) = \phi^{(k)}(b)$ for all $k \geq 1$.

Definition 1.26 (Winding number about the origin) — Assume $\phi(t) \neq 0$ for all $t \in [a,b]$. The **winding number** $w(\phi) \in \mathbb{Z}$ counts (with sign) how many times ϕ goes around the origin.
Identifying $\mathbb{R}^2 \cong \mathbb{C}$ via $z = x + iy$, one can compute

$$w(\phi) = \frac{1}{2\pi i} \oint_\gamma \frac{dz}{z},$$

where $\gamma = \phi([a,b])$.

Writing $\phi(t) = (x(t), y(t))$, one expands (as in complex analysis) to obtain

$$w(\phi) = \frac{1}{2\pi} \int_a^b \frac{x(t)y'(t) - x'(t)y(t)}{x(t)^2 + y(t)^2}\, dt.$$

If ϕ is regular and parametrized by arclength, then the unit tangent is simply $T(t) = \phi'(t)$. The **turning number** (also called the **index**) is the winding number of $T(t)$ around the origin.

Definition 1.27 (Turning number / Index) — Let $\phi : [a,b] \to \mathbb{R}^2$ be a closed regular curve parametrized by arclength. Its **index** is $\operatorname{Ind}(\phi) = w(T)$, the winding number of the unit tangent $T(t)$.

If $|\phi'(t)| = 1$, then applying the winding-number formula to $T(t) = (x'(t), y'(t))$ gives

$$w(T) = \frac{1}{2\pi} \int_a^b (x'(t)y''(t) - y'(t)x''(t))\, dt = \frac{1}{2\pi} \int_a^b \kappa(t)\, dt.$$

Theorem 1.28 If $\phi : [a,b] \to \mathbb{R}^2$ is a closed curve parametrized by arclength, then

$$\operatorname{Ind}(\phi) = \frac{1}{2\pi} \int_a^b \kappa(t)\, dt.$$

As you travel along the curve, the unit tangent $T(t)$ rotates. Curvature $\kappa(t)$ measures the *instantaneous turning rate* of this tangent (when speed is 1).

Integrating κ over the whole loop gives the *total amount of turning* the tangent has done.

The theorem states that total turning must be an integer multiple of 2π, exactly because a full rotation of a direction is 2π and the tangent must match up when the curve closes.

§2 Surfaces

§2.1 Surfaces in Euclidean space

Curves are 1-dimensional objects; locally, they look like a line segment. Surfaces are the 2-dimensional analogue: locally, they should look like a patch of the plane $\mathbb{R}^2$ sitting inside $\mathbb{R}^3$.

The formal way to capture 'looks like a plane nearby' is via **charts** (*i.e.* local parametrizations).

Definition 2.1 (Regular surface) — A subset $S \subset \mathbb{R}^3$ is a **regular surface** if for every point $p \in S$ there exist

- an open neighbourhood $V \subset \mathbb{R}^3$ of p,
- an open set $U \subset \mathbb{R}^2$,
- and a smooth map $\phi : U \to \mathbb{R}^3$

such that:

1. $\phi(U) = V \cap S$ (the map parametrizes a neighbourhood of p in the surface),
2. ϕ is a homeomorphism onto its image (so it has a continuous inverse on $\phi(U)$),
3. for every $q \in U$, the differential $d\phi_q : \mathbb{R}^2 \to \mathbb{R}^3$ is injective.

The pair (U, ϕ) is called a **chart** (or **local parametrization**) around p.

Remark 2.2 A chart is a coordinate system on the surface: you choose parameters (u, v) in an open region of $\mathbb{R}^2$ and map them smoothly into $\mathbb{R}^3$ to draw a 'patch' of the surface.

Condition (2) prevents self-overlaps or foldings at the level of topology (locally the surface really is a 2D sheet).

Example 2.3 (Graphs of functions) If $f : \mathbb{R}^2 \to \mathbb{R}$ is smooth, its graph

$$\Gamma_f = \{(x, y, f(x, y)) : (x, y) \in \mathbb{R}^2\}$$

is a regular surface. A chart is $\phi(u, v) = (u, v, f(u, v))$. In coordinates, the differential is

$$d\phi_{(u,v)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ f_x(u, v) & f_y(u, v) \end{pmatrix},$$

whose two columns are clearly linearly independent, hence $d\phi$ is injective everywhere.

This shows, particularly, that the xy-plane is a *regular surface*, for instance.

Example 2.4 (The sphere) The unit sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

is a regular surface, but it cannot be covered by one chart (any single chart would fail somewhere). A standard choice is to use **stereographic projection**, giving two charts (from the north and south poles) that together cover S^2.

Remark 2.5 What does the injectivity of $d\phi$ really mean? Writing ϕ in coordinates,

$$\phi(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix} \implies d\phi_{(u,v)} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix}.$$

This map is injective iff the two vectors $\phi_u(u, v) = \frac{\partial \phi}{\partial u}(u, v)$, $\phi_v(u, v) = \frac{\partial \phi}{\partial v}(u, v)$ are linearly independent in $\mathbb{R}^3$.

Geometrically, fixing $v = v_0$ traces a curve $u \mapsto \phi(u, v_0)$ on the surface, whose velocity at u_0 is $\phi_u(u_0, v_0)$. Similarly, fixing $u = u_0$ gives a curve with velocity $\phi_v(u_0, v_0)$. Injectivity means these two velocity directions span a *genuine plane*: the **tangent plane** to the surface at $\phi(u_0, v_0)$.

Example 2.6 (Non-example - a cone point) The cone

$$S = \{(x, y, z) : x^2 + y^2 = z^2,\ z \geq 0\}$$

is not a regular surface at the tip $(0, 0, 0)$. Away from the tip it looks smooth, but at the tip there is no single well-defined tangent plane: the surface comes in with many possible limiting tangent directions.

A huge class of surfaces arises as level sets of a function $F : \mathbb{R}^3 \to \mathbb{R}$:

$$S = F^{-1}(c) = \{p \in \mathbb{R}^3 : F(p) = c\}.$$

The key idea is: if ∇F is nonzero on the level set, then the level set is "cut out transversely" and behaves like a smooth surface.

Theorem 2.7 Let $F : \mathbb{R}^3 \to \mathbb{R}$ be smooth and let $S = F^{-1}(c)$ for some $c \in \mathbb{R}$. If $\nabla F(p) \neq 0$ for every $p \in S$, then S is a regular surface.

The proof of this theorem will require the following familiar result from analysis.

Theorem 2.8 (Inverse function theorem) Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be smooth in a neighbourhood of p, and suppose dg_p is invertible. Then there exists an open neighbourhood U of p such that

$$g : U \to g(U)$$

is a diffeomorphism (smooth with smooth inverse).

Proof. of theorem 2.7 (sketch) Fix $p \in S$. Since $\nabla F(p) \neq 0$, at least one partial derivative is nonzero at p; assume for concreteness that $F_z(p) \neq 0$. Define $g(x, y, z) = (x, y, F(x, y, z))$. Then, $dg_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ F_x(p) & F_y(p) & F_z(p) \end{pmatrix}$, whose determinant is $F_z(p) \neq 0$, so dg_p is invertible.

By the inverse function theorem 2.8, g is locally invertible near p, with smooth inverse g^{-1}.

Now notice: g sends the level set $F = c$ to the horizontal plane $\{z = c\}$, because $F(x, y, z) = c$ implies $g(x, y, z) = (x, y, c)$. So locally, the level set can be parametrized by the inverse map restricted to the plane $\{z = c\}$:

$$\phi(u, v) = g^{-1}(u, v, c).$$

This gives a smooth chart whose differential has rank 2 (because g^{-1} is a local diffeomorphism), and it parametrizes S near p. Hence S is a regular surface. □

Example 2.9 (Sphere as a level set) Take $F(x, y, z) = x^2+y^2+z^2$. Then $S^2 = F^{-1}(1)$ and

$$\nabla F(x, y, z) = (2x, 2y, 2z),$$

which is never zero on the sphere. Hence S^2 is a regular surface by theorem 2.7.

One reason graphs are so important is that *every* regular surface looks like a graph locally after a suitable choice of coordinates.

Lemma 2.10 (Local graph form) Let $S \subset \mathbb{R}^3$ be a regular surface and $p \in S$. Then in a neighbourhood of p, the surface can be written as the graph of a smooth function, in one of the forms

$$z = f(x, y), \quad y = g(x, z), \quad \text{or} \quad x = h(y, z).$$

The intuition for this theorem is: even if a surface is globally complicated, locally it is always just a smooth 'height function' above some plane (after possibly rotating axes). So the definition via charts is not abstract for its own sake: it exactly captures the idea of a smooth 2D sheet in $\mathbb{R}^3$.

§2.1.1 Charts, coordinates, and transition maps

A useful mental diagram is:

$$\begin{array}{ccc} U \subset \mathbb{R}^2 & \xrightarrow{\phi} & \phi(U) \subset S \subset \mathbb{R}^3 \\ (u, v) & \longmapsto & \phi(u, v) \end{array}$$

A regular surface $S \subset \mathbb{R}^3$ is 'locally a plane': if you zoom in near a point $p \in S$, the surface looks like a slightly bent patch of $\mathbb{R}^2$.

A **chart** makes this precise by giving you a way to use ordinary (u, v)-coordinates to *label* points on the surface near p.

Usually one chart is not enough to cover the whole surface (the sphere is the classic example). So we use many charts whose images overlap. If two charts

$$\phi : U \to S, \quad \psi : V \to S$$

overlap (so $\phi(U) \cap \psi(V) \neq \varnothing$), then points in the overlap can be described in *two* coordinate systems:

$$(u, v) \in U \quad \text{and} \quad (s, t) \in V.$$

The rule for converting coordinates is the **transition map**:

$$\psi^{-1} \circ \phi : \ \phi^{-1}(\phi(U) \cap \psi(V)) \longrightarrow \psi^{-1}(\phi(U) \cap \psi(V)).$$

This is literally 'change of coordinates on the surface'. A key idea is that all intrinsic geometry (lengths, angles, curvature, *etc.*) should not depend on which chart we use, so we must understand how quantities transform under these transition maps, *i.e.* charts are local coordinate systems on S, and transition maps are the coordinate changes.

§2.2 Tangent vectors and tangent planes

Intuitively, the tangent plane to a surface at a point p is the best linear (flat) approximation to the surface near p, just as the tangent line is the best linear approximation to a curve at a point. We now make this idea precise.

Definition 2.11 (Tangent vectors and tangent plane) — Let $S \subset \mathbb{R}^3$ be a regular surface and let $p \in S$. A **tangent vector** to S at p is a vector of the form $\alpha'(0)$, where $\alpha : (-\varepsilon, \varepsilon) \to S$ is a smooth curve with $\alpha(0) = p$.

The **tangent plane** to S at p, denoted T_pS, is the set of all tangent vectors at p:

$$T_pS := \{\alpha'(0) \mid \alpha \text{ smooth curve in } S, \ \alpha(0) = p\}.$$

As we thought of the tangent vector as the *velocity* of a particle moving along the surface and passing through p, the tangent plane is then the set of all possible velocities the particle could have at that instant — in other words, all directions in which the surface allows motion at p.

This definition does *not* mention charts, so it is intrinsic, but it is not very convenient for calculations. Charts allow us to compute T_pS explicitly.

Lemma 2.12 Let $\phi : U \to \mathbb{R}^3$ be a chart for S at p, and let $q \in U$ satisfy $\phi(q) = p$. Then,

$$d\phi_q(\mathbb{R}^2) = \operatorname{span}\left\{\frac{\partial \phi}{\partial u}(q), \ \frac{\partial \phi}{\partial v}(q)\right\} \subset T_pS.$$

Proof. Any vector in $d\phi_q(\mathbb{R}^2)$ has the form $w = a\,\frac{\partial \phi}{\partial u}(q) + b\,\frac{\partial \phi}{\partial v}(q)$ for $a, b \in \mathbb{R}$.

Consider the curve $\alpha(t) = \phi(u_0 + at, v_0 + bt)$, $q = (u_0, v_0)$. Then, $\alpha(0) = p$ and by the chain rule,

$$\alpha'(0) = a\,\frac{\partial \phi}{\partial u}(q) + b\,\frac{\partial \phi}{\partial v}(q) = w,$$

so $w \in T_pS$. □

The vectors $\partial_u \phi(q)$ and $\partial_v \phi(q)$ are the velocities of the coordinate curves on the surface. Their span is the plane generated by moving independently in the u and v directions — exactly what we expect for a tangent plane.

To prove that this description gives *all* tangent vectors, we need to show that any curve on S can locally be written in chart coordinates.

Lemma 2.13 Let $\phi : U \to \mathbb{R}^3$ be a chart for S at p. If $\alpha : (-\varepsilon, \varepsilon) \to S$ is a smooth curve with $\alpha(0) = p$, then for sufficiently small $\varepsilon' > 0$ there exist smooth functions $u(t), v(t)$ such that

$$\alpha(t) = \phi(u(t), v(t)) \quad \text{for } t \in (-\varepsilon', \varepsilon').$$

Proof. (sketch) Near p, the surface can be written as the graph of a smooth function (*e.g.* $z = f(x, y)$). This allows us to use the inverse function theorem to solve locally for the surface coordinates (u, v) as smooth functions of (x, y), and hence express any curve on the surface in chart coordinates. □

Theorem 2.14 (Tangent plane via a chart) Let $\phi : U \to \mathbb{R}^3$ be a chart for S at p, with $\phi(q) = p$. Then,

$$T_pS = d\phi_q(\mathbb{R}^2) = \operatorname{span}\left\{ \frac{\partial \phi}{\partial u}(q),\ \frac{\partial \phi}{\partial v}(q) \right\}.$$

Proof. The inclusion $d\phi_q(\mathbb{R}^2) \subset T_pS$ follows from the first lemma. Conversely, if $v = \alpha'(0) \in T_pS$, then by lemma 2.13 $\alpha(t) = \phi(u(t), v(t))$ locally, and hence

$$\alpha'(0) = \frac{\partial \phi}{\partial u}(q)\, u'(0) + \frac{\partial \phi}{\partial v}(q)\, v'(0),$$

which lies in $d\phi_q(\mathbb{R}^2)$. □

Remark 2.15 A chart does not just parametrize the surface - it also gives a concrete basis for the tangent plane via the coordinate vectors $\partial_u \phi$ and $\partial_v \phi$.

Example 2.16 Let $S = \{x^2 + y^2 + z^2 = 1\}$ and $p = (0, 0, 1)$. Writing S locally as the graph

$$\phi(u, v) = (u, v, \sqrt{1 - u^2 - v^2}),$$

we compute

$$\frac{\partial \phi}{\partial u}(0, 0) = (1, 0, 0), \quad \frac{\partial \phi}{\partial v}(0, 0) = (0, 1, 0),$$

so $T_pS = \operatorname{span}\{(1, 0, 0), (0, 1, 0)\}$, the horizontal plane $z = 0$.

If $S = F^{-1}(c)$ is a regular **level set** of a smooth function $F : \mathbb{R}^3 \to \mathbb{R}$, then the tangent plane has an especially clean description.

Proposition 2.17 Let $S = F^{-1}(c)$ be a regular level set. Then for any $p \in S$,

$$T_pS = \{\mathbf{v} \in \mathbb{R}^3 \mid \langle \mathbf{v}, \nabla F(p)\rangle = 0\} = (\nabla F(p))^{\perp}.$$

Remark 2.18 The gradient $\nabla F(p)$ points in the direction of fastest increase of F. Since F is constant along S, any motion tangent to the surface must be *orthogonal* to $\nabla F(p)$. Thus the gradient is a normal vector to the surface, and the tangent plane is its perpendicular complement.

Example 2.19 For the paraboloid $z = x^2 + y^2$, write $F(x, y, z) = x^2 + y^2 - z$. At $p = (1, 3, 10)$, $\nabla F(p) = (2, 6, -1)$, so the tangent plane is $T_pS = \{(x, y, z) \mid 2x + 6y - z = 0\}$.

§2.3 Smooth maps and differentials

A surface is locally just an open patch of $\mathbb{R}^2$ viewed inside $\mathbb{R}^3$ via a chart. So the basic idea to define smoothness on a surface, we define it in *coordinates*.

That is, we declare a map to be smooth if, after writing the surface in local parameters (u, v), the resulting coordinate expression is a smooth map between open sets in Euclidean space.

Definition 2.20 (Smooth maps between surfaces) — Let $S_1, S_2 \subset \mathbb{R}^3$ be regular surfaces. A map $F : S_1 \to \mathbb{R}^3$ is **smooth** if for every chart $\phi : U \to S_1$ the composition

$$U \xrightarrow{\phi} S_1 \xrightarrow{F} \mathbb{R}^3$$

is smooth as a map from an open set $U \subset \mathbb{R}^2$ into $\mathbb{R}^3$.

A map $F : S_1 \to S_2$ is smooth if it is **smooth** when viewed as a map $S_1 \to \mathbb{R}^3$.

A chart gives local coordinates on S_1, so $F \circ \phi$ is literally 'F written in local parameters.' Demanding that $F \circ \phi$ is smooth for every chart means the notion does not depend on any single parametrization.

On $\mathbb{R}^n$, the derivative at a point is a linear map that tells you how a function changes to first order. On a surface, we do the same: we measure first-order change *along curves* on the surface. This leads to the **differential** as a map between tangent planes.

Definition 2.21 (Differential of a smooth map) — Let $F : S_1 \to S_2$ be smooth and let $p \in S_1$. For a tangent vector $v \in T_pS_1$, choose a smooth curve $\alpha : (-\varepsilon, \varepsilon) \to S_1$ with $\alpha(0) = p$ and $\alpha'(0) = v$. Define $\beta(t) = F(\alpha(t))$, a curve in S_2 with $\beta(0) = F(p)$, and set the **differential** $dF_p(v)$,

$$dF_p(v) = \beta'(0) \in T_{F(p)}S_2.$$

Think of v as a 'direction of travel' on the surface through p. The differential dF_p tells you what velocity vector you get after applying F, *i.e.* how F pushes tangent directions forward.

Proposition 2.22 (Well-definedness) The value $dF_p(v)$ does not depend on which curve α with $\alpha(0) = p$ and $\alpha'(0) = v$ is used in definition 2.21.

Proof. Let α_1, α_2 be two such curves with the same initial position and velocity. Pick a chart $\phi : U \to S_1$ with $\phi(q) = p$. By writing $\alpha_i(t) = \phi(u_i(t), v_i(t))$ for t near 0, the chain rule gives

$$\alpha_i'(0) = \frac{\partial\phi}{\partial u}(q)\, u_i'(0) + \frac{\partial\phi}{\partial v}(q)\, v_i'(0).$$

Because ϕ is a chart, the vectors $\phi_u(q)$ and $\phi_v(q)$ are linearly independent, so $\alpha_1'(0) = \alpha_2'(0)$ forces $(u_1'(0), v_1'(0)) = (u_2'(0), v_2'(0))$. Now differentiate $F(\alpha_i(t)) = (F \circ \phi)(u_i(t), v_i(t))$ at $t = 0$ to obtain

$$(F \circ \alpha_i)'(0) = \frac{\partial(F \circ \phi)}{\partial u}(q)\, u_i'(0) + \frac{\partial(F \circ \phi)}{\partial v}(q)\, v_i'(0),$$

which is the same for $i = 1, 2$. Hence $dF_p(v)$ is independent of the chosen curve. □

Proposition 2.23 For each $p \in S_1$, the map $dF_p : T_pS_1 \to T_{F(p)}S_2$ is linear.

Proof. Take $v_1, v_2 \in T_pS_1$ and scalars c_1, c_2. In local coordinates via a chart ϕ at p, tangent vectors correspond to velocities in $\mathbb{R}^2$, and differentiation of $F \circ \phi$ gives a linear map at the point. Chasing through the definition shows $dF_p(c_1v_1 + c_2v_2) = c_1\, dF_p(v_1) + c_2\, dF_p(v_2)$. □

The most useful computational rule is: the differential sends the coordinate tangent vectors ϕ_u, ϕ_v to the corresponding partial derivatives of $F \circ \phi$. This is a kind of chain rule.

Lemma 2.24 Let $F : S_1 \to S_2$ be smooth and let $\phi : U \to S_1$ be a chart. Fix $(u_0, v_0) \in U$ and let $p = \phi(u_0, v_0)$. Then

$$dF_p\left(\frac{\partial\phi}{\partial u}(u_0, v_0)\right) = \frac{\partial(F \circ \phi)}{\partial u}(u_0, v_0), \quad dF_p\left(\frac{\partial\phi}{\partial v}(u_0, v_0)\right) = \frac{\partial(F \circ \phi)}{\partial v}(u_0, v_0).$$

Proof. Consider the curve $\alpha(t) = \phi(u_0 + t, v_0)$. Then $\alpha'(0) = \phi_u(u_0, v_0)$ and

$$dF_p(\alpha'(0)) = (F \circ \alpha)'(0) = \frac{\partial(F \circ \phi)}{\partial u}(u_0, v_0).$$

The v-formula is identical. □

We can apply the same idea to real-valued functions on a surface, $f : S \to \mathbb{R}$.

Definition 2.25 (Smooth functions and their differential) — Let $S \subset \mathbb{R}^3$ be a regular surface. A function $f : S \to \mathbb{R}$ is **smooth** if for every chart $\phi : U \to S$, the composition $f \circ \phi : U \to \mathbb{R}$ is smooth. For $p \in S$ and $v \in T_pS$, choose a curve α with $\alpha(0) = p$ and $\alpha'(0) = v$ and define the **differential** $df_p(v)$

$$df_p(v) = \frac{d}{dt} f(\alpha(t))\Big|_{t=0}.$$

Example 2.26 (A differential on the unit sphere) Let $S = \{x^2 + y^2 + z^2 = 1\}$, fix $p = (0, 1, 0)$, and define $f : S \to \mathbb{R}$ by $f(x, y, z) = z$. Since $S = g^{-1}(1)$ for $g(x, y, z) = x^2 + y^2 + z^2$, proposition 2.17 gives

$$T_pS = (\nabla g(p))^{\perp}, \quad \nabla g(p) = (0, 2, 0).$$

Hence $T_pS = \{(a, 0, c) : a, c \in \mathbb{R}\} = \operatorname{span}\{(1, 0, 0), (0, 0, 1)\}$. To compute df_p, choose a local chart ϕ near p so that $f \circ \phi(u, v) = v$. Then, by Lemma 2.24,

$$df_p\left(\frac{\partial \phi}{\partial u}(0, 0)\right) = 0, \quad df_p\left(\frac{\partial \phi}{\partial v}(0, 0)\right) = 1,$$

so df_p kills the 'x-direction' tangent and returns 1 on the 'z-direction' tangent.

There is also a surface-level analogue of the **inverse function theorem**: if dF_p is an *isomorphism*, then F is locally a *diffeomorphism*, as stated in the following proposition.

Proposition 2.27 Let $F : S_1 \to S_2$ be smooth. If $dF_p : T_pS_1 \to T_{F(p)}S_2$ is an isomorphism for some $p \in S_1$, then there exists an open neighborhood $V \subset S_1$ of p such that $F|_V : V \to F(V)$ is a diffeomorphism.

Proof. Choose charts $\phi_1 : U_1 \to S_1$ at p and $\phi_2 : U_2 \to S_2$ at $F(p)$ with $\phi_1(q_1) = p$ and $\phi_2(q_2) = F(p)$. Define the coordinate version of F by

$$g = \phi_2^{-1} \circ F \circ \phi_1 : U_1 \to U_2,$$

after shrinking U_1 so this is well-defined. By the chain rule,

$$dg_{q_1} = (d\phi_2^{-1})_{F(p)} \circ dF_p \circ (d\phi_1)_{q_1}.$$

Each factor is invertible: $d\phi_1$ and $d\phi_2$ are injective (indeed rank 2 maps giving isomorphisms $\mathbb{R}^2 \cong T_pS_1$ and $\mathbb{R}^2 \cong T_{F(p)}S_2$), and dF_p is invertible by hypothesis. Hence dg_{q_1} is invertible, so by the inverse function theorem on $\mathbb{R}^2$, g is a diffeomorphism near q_1. Conjugating back by the charts shows F is a diffeomorphism near p. □

§2.4 Normal vectors and the Gauss map

A regular surface $S \subset \mathbb{R}^3$ has a well-defined **tangent plane** T_pS at each point $p \in S$. Geometrically, there are exactly two *unit* vectors perpendicular to this plane (they point to the two possible 'sides' of the surface). Locally, one can choose one of these continuously, but globally this may fail (*e.g.* see on a Möbius strip).
If S happens to be given as a regular level set $S = F^{-1}(c) = \{p \in \mathbb{R}^3 : F(p) = c\}$, ($\nabla F(p) \neq 0$ for all $p \in S$), then proposition 2.17 tells us $T_pS = (\nabla F(p))^{\perp}$.
So $\nabla F(p)$ points in a normal direction, and a canonical unit normal is $N(p) = \frac{\nabla F(p)}{\|\nabla F(p)\|}$.
More generally, if $\varphi : U \to S$ is a chart and $p = \varphi(u, v)$, then $\frac{\partial \varphi}{\partial u}(u, v)$, $\frac{\partial \varphi}{\partial v}(u, v)$ span T_pS (theorem 2.14). Their cross product is therefore nonzero and perpendicular to T_pS, so we can define a unit normal on the chart image by

$$N(\varphi(u, v)) = \frac{\varphi_u(u, v) \times \varphi_v(u, v)}{\|\varphi_u(u, v) \times \varphi_v(u, v)\|}, \quad \text{where } \varphi_u = \frac{\partial \varphi}{\partial u},\ \varphi_v = \frac{\partial \varphi}{\partial v}$$

Intuitively, φ_u and φ_v are the velocity vectors of the two coordinate curves on the surface, so $\varphi_u \times \varphi_v$ points 'straight out of' the surface.

Definition 2.28 (Orientability and Gauss map) — A surface $S \subset \mathbb{R}^3$ is **orientable** if one can choose a unit normal vector $N(p)$ *continuously* for all $p \in S$.
Since each $N(p)$ lies on the unit sphere S^2, such a choice defines a continuous map $N : S \to S^2$, called the **Gauss map** of S.

If $S = F^{-1}(c)$ is a regular level set, then $N(p) = \nabla F(p)/\|\nabla F(p)\|$ gives a global continuous choice, so every regular level set is orientable.

Example 2.29 (Gauss map of sphere) Let $S = S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$. Take $F(x, y, z) = x^2 + y^2 + z^2$. Then $\nabla F = (2x, 2y, 2z)$, so on S^2,

$$N(x, y, z) = \frac{(2x, 2y, 2z)}{\sqrt{4(x^2 + y^2 + z^2)}} = (x, y, z).$$

Thus the Gauss map $N : S^2 \to S^2$ is the identity.

Example 2.30 (Gauss map of plane) Let $P \subset \mathbb{R}^3$ be the plane $ax + by + cz = d$ with $(a, b, c) \neq 0$. With $F(x, y, z) = ax + by + cz$, we have $\nabla F = (a, b, c)$, so

$$N(x, y, z) = \frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}},$$

a constant map. So planes have constant normal direction.

Because $N : S \to S^2$ is a smooth map between surfaces (when S is orientable and N is smooth), it has a differential

$$dN_p : T_pS \longrightarrow T_{N(p)}S^2.$$

But at the point $N(p) \in S^2$, the normal to the sphere is exactly $N(p)$ itself, so

$$T_{N(p)}S^2 = \{w \in \mathbb{R}^3 : \langle w, N(p)\rangle = 0\}.$$

On the other hand, $N(p)$ is also a normal to S at p, so

$$T_pS = \{v \in \mathbb{R}^3 : \langle v, N(p)\rangle = 0\}.$$

Hence there is a natural identification $T_{N(p)}S^2 \cong T_pS$, and we may view

$$dN_p : T_pS \to T_pS$$

as a **linear endomorphism** of the tangent plane. Intuitively, dN_p measures how quickly the normal direction changes when you move along the surface: flat surfaces have (nearly) constant normals; curved surfaces have normals that rotate as you travel.

Example 2.31 (Sphere of radius r) Let $S = \{(x, y, z) : x^2 + y^2 + z^2 = r^2\}$ with $r > 0$. Using $F(x, y, z) = x^2 + y^2 + z^2$ we get

$$N(p) = \frac{\nabla F(p)}{\|\nabla F(p)\|} = \frac{p}{r}.$$

Given any curve $\alpha(t) \in S$ with $\alpha(0) = p$ and $\alpha'(0) = v \in T_pS$,

$$dN_p(v) = \frac{d}{dt} N(\alpha(t))\Big|_{t=0} = \frac{d}{dt}\Big(\frac{\alpha(t)}{r}\Big)\Big|_{t=0} = \frac{1}{r}\alpha'(0) = \frac{1}{r}v.$$

So

$$dN_p = \frac{1}{r}\,\mathrm{Id} : T_pS \to T_pS.$$

This matches the idea that spheres have constant 'amount of bending' everywhere, and bigger spheres bend less.

Proposition 2.32 (Computing dN_p in a chart) Let $\varphi : U \to S$ be a chart and let $p = \varphi(q)$ with $q \in U$. Then dN_p is determined by

$$dN_p(\varphi_u(q)) = \frac{\partial(N \circ \varphi)}{\partial u}(q), \quad dN_p(\varphi_v(q)) = \frac{\partial(N \circ \varphi)}{\partial v}(q).$$

Proof. (sketch) By theorem 2.14, $\varphi_u(q)$ and $\varphi_v(q)$ span T_pS, so by linearity it suffices to compute dN_p on these two vectors. The identities follow directly from the chain rule lemma 2.24) applied to the map $N : S \to S^2$ composed with the chart φ. □

Remark 2.33 To gain some intuition, we can think of walking on the surface with a tiny arrow sticking straight out of the surface at our feet (the **normal**).

As we move, that arrow swivels; the Gauss map records 'which way the arrow points' and dN_p records its instantaneous rate of change in each tangent direction.

For flat planes, the arrow never swivels ($dN_p = 0$), while for spheres, the arrow swivels at a constant rate $\sim 1/r$ in every direction.

§3 The First and Second Fundamental Forms

§4 Area and Integration on Surfaces

§4.1 The Gauss-Bonnet theorem