ASYMPTOTIC METHODS

Def^n: $f(x) = O(\phi(x))$ $\xrightarrow{} $ "$\leq$" as $x \to x_0 \in \mathbb{C}$ if: f/ϕ bounded as $x \to x_0$,

i.e. $\exists M, \delta > 0$ s.t. $|f| \leq M|\phi|$ for $|x - x_0| < \delta$.

Def^n: $f(x) = o(\phi(x))$ $\to$ "$<$" as $x \to x_0 \in \mathbb{C}$ if: $f/\phi \to 0$ as $x \to x_0$,

i.e. $\forall M > 0$, $\exists \delta > 0$ s.t. $|f| \leq M|\phi|$ for $|x - x_0| < \delta$.

Def^n: $f(x) = \text{ord}(\phi)$ $\to$ "$=$" as $x \to x_0 \in \mathbb{C}$ if: f/ϕ bounded (and bounded away from zero) as $x \to x_0$, i.e. $\exists m, M, \delta > 0$ s.t. $m|\phi| \leq |f| \leq M|\phi|$ for $|x - x_0| < \delta$.

clearly, $f = o(\phi) \Rightarrow f = O(\phi)$; $f = \text{ord}(\phi) \Rightarrow f = O(\phi)$; $f = o(\phi) \Rightarrow f \neq \text{ord}(\phi)$.

N/B: for big-O, $O(\phi)^\alpha = O(\phi^\alpha)$ for $\alpha > 0$, and we can also integrate, provided integral doesn't change sign $\to$ i.e. $\int_{x_0}^{x} O(\phi)\, dx = O\left(\int_{x_0}^{x} \phi\, dx\right)$.

BUT we cannot exponentiate or differentiate.

$\hookrightarrow$ e.g. $2x = O(x)$ but $\exp(2x) \gg \exp(x)$ for $x \nearrow \infty$.

$x \sin x = O(x)$ but $x \cos x + \sin x \neq O(1)$ for $x \nearrow \infty$.

Def^n: $f(x) \sim g(x)$ as $x \to x_0$ if: $f/g \to 1$ as $x \to x_0$. (i.e. $f - g = o(g)$)

note $\to$ "$\sim$" stricter than "ord" since it distinguishes ord(1) prefactors. ($\sim \Rightarrow$ ord)

Def^n: given $\delta_0(x) \gg \delta_1(x) \gg \dots$, f has the (finite) asymptotic expansion

$f(x) \sim \sum_{n=0}^{N} a_n \delta_n(x)$ if $f(x) = \sum_{n=0}^{N} a_n \delta_n(x) + o(\delta_N(x))$ $\to$ error is smaller than last term kept.

N/B: for infinite expansion ($N = \infty$) and for any partial sum ($\forall N$), the error is smaller than the last term kept.

POWER SERIES METHOD $\to$ for solving eq^{n}s use (naive) $x \sim \sum_{n=0}^{\infty} x_n \varepsilon^n$

substitute into eq^n and solve for x_n for each order $O(\varepsilon^0)$, $O(\varepsilon^1)$, ... etc.

ITERATION METHOD $\to$ more general approach even if x not power series.

Write eq^n into form $x = f_\varepsilon(x)$ where f_ε does not depend on x at leading order.

Start with evaluating $x = f_\varepsilon(x)$ at leading order i.e. $x = f_\varepsilon(x) = A + O(\varepsilon)$.

Then consider $x = f_\varepsilon(A + O(\varepsilon))$ and repeat etc.

note → not necessary to gain one power of ε of accuracy in each iteration.

If x is exact sol^n and x_n is approximation, introduce error $E_n = x_n - x$

so applying funct^n once → $E_{n+1} = f_\varepsilon(x + E_n) - x \approx f_\varepsilon(x) + f_\varepsilon'(x) E_n - x$

$$= f_\varepsilon'(x) E_n$$

↳ so if $f_\varepsilon'(x) = O(\varepsilon)$, error shrinks by ε at each iteration.

Method does not work if $f_\varepsilon'(x)$ is not small.

<u>METHOD OF DOMINANT BALANCE</u> → e.g. consider $\varepsilon x^2 + (\cos\varepsilon) x - e^\varepsilon = 0$ $(\varepsilon \searrow 0)$

Then at leading order we have $x \approx 1$ <u>but</u> ε multiplies highest degree term so we lose a sol^n. To make quadratic term important, expect x larger than $O(1)$ as $\varepsilon \searrow 0$ and seek rescaling $\underline{x = \delta(\varepsilon) X}$ $(\delta >> 1,\ X = \text{ord}(1))$

$$\Rightarrow \underset{\text{ord}(\varepsilon\delta^2)}{\underset{\downarrow}{\varepsilon\delta^2 X^2}} + \underset{\text{ord}(\delta)}{\underset{\downarrow}{(\cos\varepsilon)\delta X}} - \underset{\text{ord}(1)}{\underset{\downarrow}{e^\varepsilon}} = 0$$

→ impossible for just one term to be largest since then that term = 0, so require at least 2 to be largest at same time.

correct balance is $\text{ord}(\varepsilon\delta^2) = \text{ord}(\delta) \Rightarrow \delta = \varepsilon^{-1}$. Then substitute $x = \varepsilon^{-1} X$ and solve corrections w/ power series ansatz $X \sim \sum_{n=0}^{\infty} X_n \varepsilon^n$.

To find corrections for $x \approx 1$ sol^n, use $x = 1 + \tilde{x}_1$ $(\tilde{x}_1 << 1)$ and solve for $\tilde{x}_1$ dropping higher order errors

<u>Transcendental (non-poly) eq^{n}s</u> → draw a sketch to obtain guess for leading order value of root.

e.g. INVERTING A FUNCT^n: find expansion $x = f^{-1}(\varepsilon)$ for $f(x) = x^3 e^{-x}$, near $f(0)=0$

near $x = 0$, $\varepsilon = f(x) = x^3 - x^4 + \frac{x^5}{2} + O(x^6) \approx x^3 \Rightarrow x \approx \varepsilon^{1/3}$.

for corrections, use iteration → $x = \left(\varepsilon + x^4 - \frac{x^5}{2} + O(x^6)\right)^{1/3}$ → rearranging the above w/ x^3

$$= \varepsilon^{1/3}\left[1 + \frac{x^4}{\varepsilon} + O\left(\frac{x^5}{\varepsilon}\right)\right]^{1/3} = \varepsilon^{1/3}\left(1 + O(\varepsilon^{1/3})\right)$$

for other sol^{n}s $x^3 e^{-x} = \varepsilon$, sol^n should exist for $x \nearrow \infty$ since exponential dominates so set up iteration $x = 3\ln x - \ln\varepsilon$ $\left(x \sim \ln\left(\frac{1}{\varepsilon}\right)\right)$.

e.g. $\underline{\ln x + e^{1/x} = y}$ for $y \nearrow \infty$. 2 sol^{n}s since $\ln x$ dominant for large x, and $e^{1/x}$ dominant for small x.

for small sol^n, solve for x in exponent: $x = \dfrac{1}{\ln(y - \ln x)} \underset{y >> \ln x}{=} \dfrac{1}{\ln(y + O(\ln\ln y))} = \dfrac{1}{\ln y + O\left(\frac{\ln\ln y}{y}\right)}$ → $\dfrac{1}{\ln y} + O\left(\dfrac{\ln\ln y}{y(\ln y)^2}\right)$

for large sol^n, $x = \exp\{y - e^{1/x}\} = \exp\{y - 1 + O(1/x)\}$

$$= \exp\{y-1\}\left(1 + O\left(\tfrac{1}{e^y}\right)\right) = \exp\{y-1\} + o(1)$$

<u>TERM BY TERM INTEGRATION.</u> e.g. $I(x) = \int_x^1 \frac{\ln(1+t)}{t^3}\,dt$ as $x \searrow 0$.

expand integrand $\frac{\ln(1+t)}{t^3} = \frac{1}{t^2} - \frac{1}{2t} + \frac{1}{3} + O(t)$ as $t \searrow 0$. But expansion only works for small t so want to integrate $\int_0^t$.

$\therefore$ separate out singular terms + want to integrate term by term $\int_0^t$ non-singular.

$$\Rightarrow I(x) = \int_x^1 \underbrace{\frac{\ln(1+t)}{t^3} - \frac{1}{t^2} + \frac{1}{2t}}_{\text{let } = g(t)}\,dt + \underbrace{\int_x^1 \frac{1}{t^2} - \frac{1}{2t}\,dt}_{}$$

this can be integrated directly

then $\int_x^1 g(t)\,dt = \underbrace{\int_0^1 g(t)\,dt}_{\text{constant}} - \int_0^x g(t)\,dt$ and integrate term by term.

e.g. for $E_1(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt$ as $x \nearrow \infty$ → integrate by parts → higher + higher powers of t in denominator

$$\Rightarrow E_1(x) = \left[e^{-t}\left(\tfrac{1}{t}\right)\right]_x^\infty - \int_x^\infty \frac{e^{-t}}{t^2} = e^{-x}\left[\frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} + \dots \frac{(-1)^N (N-1)!}{x^N}\right] + \underbrace{(-1)^N N! \int_x^\infty \frac{e^{-t}}{t^{N+1}}\,dt}_{R_N}$$

<u>SPLITTING RANGE OF INTEGRATION</u> e.g. $I(\varepsilon) = \int_0^1 \frac{dx}{(1+x)(\varepsilon+x)^{1/2}}$ as $\varepsilon \searrow 0$.

expanding for small ε, $I(\varepsilon) = \int_0^1 \frac{1}{(1+x)^{1/2}}\left(1 + \frac{\varepsilon}{x}\right)^{1/2} dx$ but this only valid for ε/x small so breaks down at $x = \text{ord}(\varepsilon)$.

First term integrates to $1 = \frac{\pi}{2}$ ↗.

we have global scale $x = \text{ord}(1)$ and local scale $x = \text{ord}(\varepsilon)$ so <u>idea</u> is to split integral into intermediate scale δ, $\varepsilon \ll \delta \ll 1$.

$\therefore I = \int_0^\delta + \int_\delta^1$. So for second integral global contribution valid so can expand fully with ε/x assumed small.

For first integral, rescale $x = \varepsilon X$ and so can integrate w/ expanding $\frac{1}{1+\varepsilon X}$.

N/B: all terms depending on δ should cancel (as does not depend where we split).

e.g. $I(\varepsilon, \alpha) = \int_0^1 \frac{dx}{(1+x)(\varepsilon+x)^\alpha}$ for $\alpha > 0$, $\varepsilon \searrow 0$.

at $x = \text{ord}(1)$ → integrand $\sim \frac{1}{(1+x)x^\alpha}$ → ord(1)

at $x = \text{ord}(\varepsilon)$ → integrand $\sim \frac{1}{(\varepsilon+x)^\alpha}$ → ord($\varepsilon^{-\alpha}$)

$\therefore$ if $0<\alpha<1$ global (ord(1)) dominates but if $\alpha>1$ then local (ord($\varepsilon^{-\alpha}$)) dominates.

at leading order, $I \sim \int_0^1 \frac{dx}{(1+x)x^\alpha}$ for $0<\alpha<1$ and

$I \overset{x=\varepsilon X}{\sim} \varepsilon^{-\alpha+1} \int_0^\infty \frac{dX}{(1+X)^\alpha}$ for $\alpha > 1$. For obtaining correction terms, use method of splitting the range.

GAMMA FUNCTⁿ $\rightarrow$ $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$ (w/ $\Gamma(x+1) = x \cdot \Gamma(x)$)

Watson's lemma: if $f(t) \sim \sum_{n=0}^{\infty} a_n t^{\alpha_n}$ as $t \searrow 0$ for $-1 < \alpha_0 < \alpha_1 < \dots$

then following integral has asymptotic expansion: $\boxed{\int_0^T e^{-xt} f(t)\, dt \sim \sum_{n=0}^{\infty} a_n \frac{\Gamma(\alpha_n+1)}{x^{\alpha_n+1}}}$

as $x \nearrow \infty$ (and N/B: $0 < T \leq \infty$).

↳ i.e. lemma says substituting expansion for f + integrating term by term yields correct answer

Generally if $\int_a^b e^{xh(t)} f(t)\, dt$ then use substitutⁿ $h(t) = h(a) - s$ to apply Watson's.

LAPLACE'S METHOD: integrals of form $I(x) = \int_a^b e^{xh(t)} \cdot f(t)\, dt$ as $x \nearrow \infty$.

↳ principle same as Watson's $\rightarrow$ expand $h(t)$ in exponent and dominant contribution comes from small region near maximum of h (either endpoint or stationary point $h'=0$, $h''<0$)

Steps $\rightarrow$ identify max $h(t)$, t_* and use $\int_{-\delta}^{\delta} ds$ (or $\int_0^\delta$ or $\int_{-\delta}^0$ if t_* on boundary)

where $s = t - t_*$.

↳ expand h and f for small s.

↳ rescale s to make first varying term ord(1), e.g. for $\exp\{-\frac{xs^2}{2}\}$, let $u^2 = \frac{xs^2}{2}$.

↳ keep ~~these~~ this ~~terms~~ term in exponent + multiply w/ expansion of f.

↳ expand any higher order terms in the exponential (not "u^2") and use Γ to solve.

e.g. MOVING MAXIMUM $\rightarrow$ behaviour of $n! = \Gamma(n+1) = \int_0^\infty e^{-t} t^n\, dt$ for $n \nearrow \infty$

writing $t^n = e^{n \ln t}$ but $h(t) = \ln t$ grows indefinitely so seek maximum of integrand.

$e^{-t} t^n = e^{-t + n \ln t}$ which maximises at $t_* = n \rightarrow$ moving.

$\therefore$ rescale as $t = ns$ to obtain exponent which attains max. at fixed $s = 1$

Riemann-Lebesgue Lemma: if $\int_a^b |f(t)|\, dt$ exists, then

$$\int_a^b e^{ixt} f(t)\, dt \rightarrow 0 \quad \text{as } x \nearrow \infty.$$

↳ oscillatory integral.

METHOD OF STATIONARY PHASE, $\rightarrow$ apply to $I(x) = \int_a^b e^{ix\varphi(t)} f(t)\, dt$, $x \nearrow \infty$

$\varphi(t)$ is phase + dominant contribution comes from near where φ stationary.

↳ N/B: only valid to leading order

Mostly, the method of stationary phase needing to evaluate following integral;

$$\int_0^\infty e^{iu^p}\,du \overset{v=u^p}{=} \frac{1}{p}\int_0^\infty e^{iv}\, v^{\frac{1}{p}-1}\,dv$$ → this requires contour deformation and want to make e^{iv} into e^{-w} to apply Γ.

∴ rotate real contour $[0,\infty]$ to imaginary axis using sub $v = iw$. $(= e^{i\frac{\pi}{2}}w)$

$$\Rightarrow \frac{1}{p} e^{i\frac{\pi}{2p}} \int_0^\infty e^{-w} w^{\frac{1}{p}-1}\,dw = \frac{1}{p} e^{i\frac{\pi}{2p}} \cdot \Gamma(1/p).$$

(N/B:  → contribution from arc vanishes.)

METHOD OF STEEPEST DESCENT → consider $I(x) = \int_C e^{xh(t)} f(t)\,dt$, $x \nearrow \infty$

where $x \in \mathbb{R}$, but $h(t)$ complex, and C is a complex contour.

Idea is to deform original contour to a new contour (w/ same endpoints) so that we obtain an integral exactly equal to original one, but can also be approximated using Laplace's method.

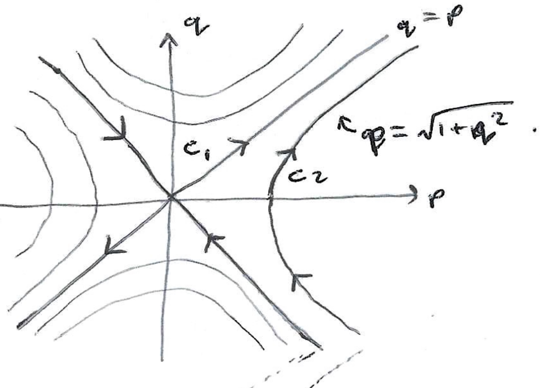
- For SD between ENDPOINTS, SD contours are where $h(t)$ has CONSTANT IMAGINARY PART.

e.g. $I(x) = \int_0^1 e^{ixt^2}\,dt$ as $x \nearrow \infty$ so let $t = p + iq$ and consider $h(t)$,

$h(t) = it^2 = i(p+iq)^2 = -2pq + i(p^2 - q^2)$

Since require $\mathrm{Im}(h)$ constant on SD and we have endpoints $(p,q) = (0,0) \to (1,0)$

then $p^2 - q^2 = 0 \Rightarrow p = \pm q$ and $p^2 - q^2 = 1 \Rightarrow p = \pm\sqrt{1+q^2}$.

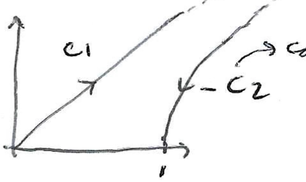


these all show the SD contours where $\mathrm{Im}(h) = \text{const}$.

N/B: arrows point in the direction that $\mathrm{Re}(h) = -2pq$ decreases.

MUST SELECT CONTOURS WHICH DESCEND FROM THE ENDPOINTS ∴ pick $\underset{(t=0)}{q = p}$ and $\underset{(t=1)}{p = \sqrt{1+q^2}}$.

these contours "meet" at "$e^{i\frac{\pi}{4}} \cdot \infty$".

∴ 

→ call this $-C_2$ to keep things +ve.

$$\therefore I(x) = \int_{C_1} - \int_{C_2}.$$

on C_1, $t = p + ip$

on C_2, $t = p + i\sqrt{p^2-1}$

$\Rightarrow$ for C_1, $dt = (1+i)\,dp \to \int_{C_1} = \int_{p=0}^\infty e^{-2xp^2}(1+i)\,dp$ → apply Laplace's here w/ max. at $p = 0$.

for C_2, $dt = \left(1 + \frac{ip}{\sqrt{p^2-1}}\right)dp \to \int_{C_2} = \int_{p=1}^\infty e^{ix} e^{-2xp\sqrt{p^2-1}}\left(1 + \frac{ip}{\sqrt{p^2-1}}\right)dp$.

- For SD with saddle points → ~~need to understand~~ previously, following the SD contour "downhill" resulted in contours converging to same infinite direction, but generally we may end up at different directions/locations at ∞ → so how to connect?

To connect 2 endpoints at 2 infinitys, we need a SD contour that passes over a <u>SADDLE POINT</u> (i.e. where $h'(t_*) = 0$).

N/B: this saddle needs to have <u>maximum $\mathrm{Re}(h)$</u>.

$\Rightarrow$ we need to consider contours near a saddle point.

For n-th order saddle, $h^{(n)}(t_*) = Ae^{i\alpha}$ $(A>0)$, and SD directions are found by seeking $\boxed{\pi + 2\pi k = \alpha + n\arg(s)}$ $\rightarrow k = 0, 1, \dots, n-1$.

N/B: the maximum of $\mathrm{Re}(h)$ on the contour must be attained at an endpoint or saddle point, while moving in SD direction.

<u>ODEs</u> $\rightarrow$ linear eq^{n}s w/ algebraic behaviour ($y = cx^\alpha$) has $\frac{d}{dx} = \mathrm{ord}\left(\frac{1}{x}\right)$ ↳ not valid for differentiating large funct^{n}s e.g. exponential.

e.g. $y'' - xy = e^{-x}$ as $x \to 0$ $\Rightarrow$ must add a particular sol^n to a sol^n of homogenous eq^n $y'' - xy = 0$.

↳ using $\frac{d}{dx} = \mathrm{ord}\left(\frac{1}{x}\right)$, expect $y'' \gg xy$ so dom. balance $\rightarrow y'' \approx 1$.

so $y \approx \frac{1}{2}x^2$ is particular sol^n and use $y = \frac{1}{2}x^2 + \tilde{y}_1$ for corrections.

for homogenous sol^n, dom. balance gives $y'' \approx 0 \rightarrow$ 2 indep. sol^{n}s, $y \approx A$ and $y \approx Bx$.

<u>EQUIDIMENSIONAL FORM</u> $\rightarrow P_n(x)\underbrace{(x^n y^{(n)})}_{\mathrm{ord}(y)} + P_{n-1}\underbrace{(x^{n-1}y^{(n-1)})}_{\mathrm{ord}(y)} + \dots + P_1 \underbrace{xy'}_{\mathrm{ord}(y)} + P_0 y = f(x)$

↳ this reveals the terms which $P_i(x)$ are dominant in the limit ($x \to 0$ or $x \to \infty$) are featured in the dominant balance.

N/B: in a linear case, can try power law ansatz $y = Ax^\alpha$ ↗ and solve an eq^n for α to get sol^{n}s.

If highest order derivative not part of dom. balance $\rightarrow$ lose sol^{n}s.

(e.g. $y'' - xy = 0$ as $x \to \infty$) so IDEA is to make derivatives become larger.

<u>So</u> use $y = e^{S(x)}$ $\rightarrow y'' = (S'^2 + S'')e^S$ (and we can assume $S'^2 \gg S''$) since want $S' \gg 1/x$

N/B: $S \sim S_0$ does not imply $e^S \sim e^{S_0}$ so want next correction too.

e.g. $y'' - xy = 0$ as $x \to \infty$, $y = e^S \Rightarrow S'^2 + S'' - x = 0 \Rightarrow S'^2 \approx x$.

so $S \approx \pm\frac{2}{3}x^{3/2} = S_0$ then $S = S_0 + \tilde{S}_1 \Rightarrow \cancel{S_0'^2} + 2S_0'\tilde{S}_1' + \tilde{S}_1'^2 + S_0'' + \tilde{S}_1'' - \cancel{x} = 0$

and estimate $\tilde{S}_1'^2 \ll S_0'\tilde{S}_1'$ and $\tilde{S}_1'' \ll S_0''$ (since correction $\tilde{S}_1 \ll S_0$).

If we have an ODE w/ a small parameter (ε)

↳ then we may consider power series expansion $y \sim y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots$

MATCHED ASYMPTOTIC EXPANSIONS

e.g. (basic example) $\varepsilon y'' + y' + y = 0$ with $y(0) = -1$, $y(1) = e^{-1}$ as $\varepsilon \searrow 0$.

we seek expansion $y \sim y_0 + \varepsilon y_1 + \dots$ but notice both BCs cannot be satisfied together.

$\therefore$ call $y \sim y_0 + \varepsilon y_1 + \dots$ OUTER SOL^n so at leading order:

$y_0' + y_0 = 0 \Rightarrow y_0 = C_0 e^{-x}$; at $O(\varepsilon) \to y_1' + y_1 = -C_0 e^{-x}$

$\Rightarrow y_1 = -C_0 x e^{-x} + C_1 e^{-x}$.

so $y \sim y_0 + \varepsilon y_1$, and we seek location of boundary layer for INNER SOL^n.

take scaling $x = x_0 + \delta(\varepsilon) X$ (where $X = \text{ord}(1)$, $\delta \ll 1$), $y(x) = Y(X)$

only works for linear problems

$\Rightarrow \frac{\varepsilon}{\delta^2} Y'' + \frac{1}{\delta} Y' + Y = 0$ so $\text{ord}\left(\frac{\varepsilon}{\delta^2}\right) = \text{ord}\left(\frac{1}{\delta}\right)$ for dom. balance $\Rightarrow$ take $\delta = \varepsilon$

$\therefore$ leading order inner sol^n $\to Y'' + Y' + \varepsilon Y = 0 \Rightarrow Y_0 = A + Be^{-X}$.

Notice as $X \searrow -\infty$, Y_0 diverges so we cannot match to an outer sol^n on the left. Also $X \nearrow \infty$, Y_0 does not blow up so we match to right side $\Rightarrow$ outer is on the right + BL is at $x = 0$ $\therefore$ $x = \varepsilon X$.

outer sol^n satisfies BCs at $x = 1 \to C_0 = C_1 = 1$ so $y \sim e^{-x} + \varepsilon(-xe^{-x} + e^{-x})$

Now MATCHING $\to$ we want: $\lim_{X \nearrow \infty} Y_0 = \lim_{x \searrow 0} y_0 \Rightarrow -B - 1 = 1 \Rightarrow B = -2$

[APPLY BC at $x = 0$ to INNER $\Rightarrow A + B = -1 \Rightarrow Y_0 = Be^{-X} - B - 1$]

since we have y_1, find INNER to next order as well. (to obtain composite)

$Y_1'' + Y_1' = -Y_0 = -1 + 2e^{-X}$ (and $Y_1(0) = 0$)

$\Rightarrow Y_1 = -X - 2Xe^{-X} + Ce^{-X} + D = -X - 2Xe^{-X} + Ce^{-X} - C$.

- $y(\varepsilon X) = e^{-\varepsilon X} + \varepsilon[e^{-\varepsilon X} - \varepsilon X e^{-\varepsilon X}] + O_x(\varepsilon^2)$ indicating variable "x" held fixed in limit $\varepsilon \searrow 0$.

$\approx 1 - \varepsilon X + \varepsilon + O_x(\varepsilon^2) + O_x(\varepsilon^2)$

- $Y(x/\varepsilon) = 1 - 2e^{-x/\varepsilon} + \varepsilon\left(-x/\varepsilon - 2\frac{x}{\varepsilon}e^{-x/\varepsilon} + Ce^{-x/\varepsilon} - C\right) + O_x(\varepsilon^2)$

$\approx 1 + O_x(\varepsilon^2) + \varepsilon(-x/\varepsilon + O_x(\varepsilon) + O_x(\varepsilon^2) - C) + O_x(\varepsilon^2)$

$= 1 - x - C\varepsilon + O_x(\varepsilon^2) + O_x(\varepsilon^2) \Rightarrow C = -1$

$\therefore$ overlap region is $\to$ $\underline{1 - x + \varepsilon}$.

$\therefore$ composite : $y \sim \left(e^{-x} + \varepsilon(e^{-x} - xe^{-x})\right) + \left(1 - 2e^{-x/\varepsilon} - x + \varepsilon - 2xe^{-x/\varepsilon} - \varepsilon e^{-x/\varepsilon}\right) - (1 - x + \varepsilon)$.

e.g. (advanced example) $y'' = -6x + \frac{\varepsilon}{y^4}$, $y(0) = 1$, $y'(0) = 0$.

seek outer sol^n $\rightarrow$ $y_0'' = -6x$ $\rightarrow$ $y_0 = -x^3 + Ax + B$ so $A = 0$, $B = 1$ w/ BCs.

at next order $\rightarrow$ $y_1'' = y_0^{-4} = (1 - x^3)^{-4}$ but blows up at $x = 1$.

$\therefore$ as $x \nearrow 1$ correction εy_1 eventually becomes larger than y_0

$\Rightarrow$ require BL (or interior layer) near $x = 1$. So seek $x = 1 + \delta_x(\varepsilon) X$

and $y(x) = \delta_y(\varepsilon) Y(X)$ (but 2 unknowns δ_x, δ_y and only 1 dom. balance)

so also use local behaviour of outer sol^n $\rightarrow$ $y_0 = \text{ord}(1 - x)$ as $x \nearrow 1$

$\Rightarrow$ ord(y_0) = ord$(\delta_y Y)$ = ord$(\delta_x X)$ = ord$(1 - x)$ $\Rightarrow$ $\delta_y = \delta_x$.

$$\therefore \underbrace{\frac{\delta_y}{\delta_x^2} Y''}_{\text{ord}(1/\delta_x)} = \underbrace{-6(1 + \delta_x X)}_{\text{ord}(1)} + \underbrace{\frac{\varepsilon}{\delta_y^4} Y^{-4}}_{\text{ord}(\varepsilon/\delta_x^4)} \longrightarrow \frac{1}{\delta_x} = \frac{\varepsilon}{\delta_x^4} \Rightarrow \delta_x = \varepsilon^{1/3}$$

[Alternatively, one could argue asymptotic ORDERING BREAKS DOWN when ord(y_0) = ord(εy_1) $\Rightarrow$ near $x = 1$, ord$(1 - x)$ = ord$(\varepsilon(1 - x)^{-2})$ $\Rightarrow$ $\underbrace{1 - x}_{y_0}$ = ord$(\varepsilon^{1/3})$].

so inner eq^n : $Y'' = -6\varepsilon^{1/3}(1 + \varepsilon^{1/3} X) + Y^{-4}$ $\Rightarrow$ $Y_0'' = Y_0^{-4}$ at leading order.

note since outer behaviour $y \sim 1 - (1 + \varepsilon^{1/3} X)^3 \sim -3\varepsilon^{1/3} X$ as $x \nearrow 1$, $Y_0 \sim -3X$ as $X \searrow -\infty$ to match.

N/B: INTERIOR LAYER, may appear when eq^n has singularity or coefficient in eq^n vanishes.

WKB METHOD (away from turning pts) $\varepsilon^2 y'' - q(x) y = 0$ as $\varepsilon \searrow 0$.

cannot simply balance two terms w/ $x = \varepsilon X$ since ~~scaling~~ expansion not valid for $X = \text{ord}\left(\frac{1}{\varepsilon}\right)$.

$\therefore$ use change of variables $y = e^{S(x)/\delta}$ $\Rightarrow$ $\varepsilon^2 \left(\frac{S'^2}{\delta^2} + \frac{S''}{\delta}\right) - q(x) = 0$

since $\frac{1}{\delta^2} \gg \frac{1}{\delta}$, one can obtain scaling w/ $\delta = \varepsilon$ for balance.

$\Rightarrow$ $S'^2 + \varepsilon S'' = q(x)$. So $S_0'^2 = q$, $2S_0'S_1' + S_0'' = 0$

$\Rightarrow$ $S_0' = \pm q^{1/2}$, $S_1' = -\frac{S_0''}{2S_0'} = \left(-\frac{1}{2}\ln S_0'\right)' = \left(-\frac{1}{4}\ln q\right)'$

$\therefore$ $y_\pm = \frac{1}{q^{1/4}} \exp\left\{\pm \frac{1}{\varepsilon} \int q^{1/2}\, dx\right\}$ and $y \sim A_+ y_+ + A_- y_-$

N/B: for $q < 0$ write $\rightarrow$ $y_\pm = \frac{1}{(-q)^{1/4}} \exp\left\{\pm \frac{i}{\varepsilon} \int (-q)^{1/2}\, dx\right\}$. can be written in sum of sin + cos too.

WKB METHOD (near turning pt) $\quad \varepsilon^2 y'' - q(x) y = 0. \quad (\varepsilon \searrow 0)$

when $q < 0$ sol^{n}s are oscillatory, when $q > 0$ sol^{n}s are exponential.

Sol^{n}s given as previously, but when $q = 0$ sol^{n}s blow up.

Suppose $q = 0$ at $x = 0$ (and $q < 0$ for $x < 0$, $q > 0$ for $x > 0$).

Assume $q'(0) = m$ so we can expand $q \to q(x) \approx mx$ near $x = 0$.

$\Rightarrow \varepsilon^2 y'' - mxy \approx 0 \to$ rescale $x = \underbrace{\varepsilon^{2/3} m^{-1/3}}_{\delta(\varepsilon)} X$, $y = Y \Rightarrow \underbrace{Y'' - XY = 0}_{\text{Airy eq}^n}$

so $Y \sim C \cdot Ai(X) + D \cdot Bi(X)$

Match to outer sol^n on right w/ $X \nearrow \infty$ and left outer sol^n w/ $X \searrow -\infty$.

expand the outer sol^{n}s near $x = 0$ using $q \approx mx$ + evaluate $\int q^{1/2} dx$, $q^{1/4}$ etc.

PERTURBED HARMONIC OSCILLATORS perturbation of $y'' + y = 0$.

POINCARÉ-LINDSTEDT / METHOD of STRAINED COORDINATES $\to$ idea is to use angular frequency $\omega = 1 + \omega_1 \varepsilon + O(\varepsilon^2)$ to introduce 'strained' time $s = \omega t$.

↳ if we consider $y'' + y - \varepsilon y^3 = 0$ then seeking sol^n $y = y_0 + \varepsilon y_1 + \dots$ we obtain secular term $\propto e^{it}$ at $O(\varepsilon)$ so expansion invalid for large $t = \text{ord}(1/\varepsilon)$.

$\frac{d}{dt} = \omega \frac{d}{ds} \Rightarrow \omega^2 y'' + y - \varepsilon y^3 = 0 \Rightarrow y_0'' + y_0 = 0$

$\Rightarrow y_1'' + y_1 = y_0^3 - 2\omega_1 y_0''$ $\quad$ since y is 2π periodic in s.

to determine $\omega_1 \to$ we require secular terms ($\propto e^{it}$) to cancel out.

METHOD OF MULTIPLE SCALES $\to$ introduce a slow time $\to T = \varepsilon t$, and treat it separately to t so $y = y(t, T) \to \frac{d}{dt} = \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T}$ so ODE into PDE.

when solving at leading order remember $\partial_t^2 y_0 + y_0 = 0 \Rightarrow y_0 = \underbrace{A_0(T)}_{\text{constant now func}^n \text{ of } T} e^{it} + cc$

going to next order y_1 gives conditi^n for NO secular term

↳ this gives ODE for $A_0(T)$. Often useful to write $A_0(T) = r(T) e^{i\phi(T)}$

so we get 2 ODEs for r and $\phi \to$ comparing Re and Im parts.

N/B: for expansion to be valid at $T = \text{ord}(1/\varepsilon) \Rightarrow t = \text{ord}(1/\varepsilon^2)$ we need to introduce slow-slow time $T_2 = \varepsilon^2 t \Rightarrow \frac{d}{dt} = \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T} + \varepsilon^2 \frac{\partial}{\partial T_2}$.

N/B: $y_0'' + y_0 = 0 \Rightarrow y_0 = A(T) e^{it} + cc = 2[\text{Re}(A) \cos t - \text{Im}(A) \sin t]$ [illegible]

↳ $= 2|A| \cos(t + \arg A)$